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Abstract. We obtain new oscillation and gradient bounds for the viscosity solutions of fully nonlinear degenerate elliptic equations where the Hamiltonian is a sum of a sublinear and a superlinear part in the sense of Barles and Souganidis (2001). We use these bounds to study the asymptotic behavior of weakly coupled systems of fully nonlinear parabolic equations. Our results apply to some "asymmetric systems" where some equations contain a sublinear Hamiltonian whereas the others contain a superlinear one. Moreover, we can deal with some particular case of systems containing some degenerate equations using a generalization of the strong maximum principle for systems.

1. Introduction

One of the main result of this work is to obtain new results about the large time behavior of the solution $u(x,t) = (u_1(x,t), \cdots, u_m(x,t))$ of the fully nonlinear parabolic system

$$(1.1) \quad \begin{cases} \dfrac{\partial u_i}{\partial t} + \sup_{\theta\in\Theta} \left\{ -\mathrm{trace}(A_{\theta i}(x) D^2 u_i) + H_{\theta i}(x, Du_i) \right\} + \displaystyle\sum_{j=1}^{m} d_{ij} u_j = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \mathbb{T}^N \times (0,+\infty), \quad 1 \le i \le m, \\ u_i(x,0) = u_{0i}(x), \qquad x \in \mathbb{T}^N, \end{cases}$$

in the periodic setting ($\mathbb{T}^N$ denotes the flat torus $\mathbb{R}^N/\mathbb{Z}^N$), where the equations are linearly coupled through a matrix $D = (d_{ij})_{ij}$ which is assumed to be monotone and irreducible. The set Θ is a metric space, the diffusion matrices can be written $A_{\theta i}(x) = \sigma_{\theta i}(x)\sigma_{\theta i}(x)^T$ with $\sigma_{\theta i}$ bounded Lipschitz continuous and H_i, u_{0i} are continuous.

To simplify the presentation, we present our results in the simplest case without dependence with respect to θ and for $m = 2$. See Section 3.5 for some discussions about the general case. We then consider

$$(1.2) \quad \begin{cases} \dfrac{\partial u_1}{\partial t} - \mathrm{trace}(A_1(x) D^2 u_1) + H_1(x, Du_1) + u_1 - u_2 = 0, \\ \dfrac{\partial u_2}{\partial t} - \mathrm{trace}(A_2(x) D^2 u_2) + H_2(x, Du_2) + u_2 - u_1 = 0, \quad (x,t) \in \mathbb{T}^N \times (0,+\infty), \\ u_1(x,0) = u_{01}(x), \ u_2(x,0) = u_{02}(x) \qquad x \in \mathbb{T}^N, \end{cases}$$

Date: May 15, 2015.

2010 *Mathematics Subject Classification.* Primary 35J70; Secondary 35K40, 35B40, 35B45, 35B50 35F21, 49L25.

Key words and phrases. Nonlinear degenerate parabolic equations, Nonlinear degenerate elliptic equations, Hamilton-Jacobi equations, monotone systems, gradient bounds, oscillation, strong maximum principle, ergodic problem, asymptotic behavior, viscosity solutions.

The precise assumptions on the Hamiltonians H_i will be explained below.

We prove the following asymptotic behavior of the solution,

$$u_i(x,t) + c_i t \to v_i(x) \quad \text{uniformly in } \mathbb{T}^N \text{ as } t \to \infty, \text{ for } i = 1,2, \tag{1.3}$$

where $c = (c_1, c_2) \in \mathbb{R}^2$ and $v = (v_1, v_2) \in W^{1,\infty}(\mathbb{T}^N)^2$ are solutions of the so-called ergodic problem

$$\begin{cases} -\mathrm{trace}(A_1(x)D^2 v_1) + H_1(x, Dv_1) + v_1 - v_2 = c_1, \\ -\mathrm{trace}(A_2(x)D^2 v_2) + H_2(x, Dv_2) + v_2 - v_1 = c_2, \quad x \in \mathbb{T}^N. \end{cases} \tag{1.4}$$

Let us give immediately one of the most striking application of our result. We are able to prove the convergence for *asymmetric systems* like

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \mathrm{trace}(A_1(x)D^2 u_1) + \langle b_1(x), Du_1 \rangle + \ell_1(x) + u_1 - u_2 = 0, \\ \dfrac{\partial u_2}{\partial t} - \mathrm{trace}(A_2(x)D^2 u_2) + |Du_2|^2 + \ell_2(x) - u_1 + u_2 = 0, \end{cases} \tag{1.5}$$

where A_1 is uniformly elliptic and A_2 may be degenerate. The name asymmetric means that different equations can have different natures as above: the first one contains a sublinear Hamiltonian and is uniformly elliptic whereas the second one contains a superlinear one and may be degenerate. The general framework is presented below. To explain the main difficulty to prove this result, let us recall the related results for scalar equations.

To study the large time behavior for parabolic nonlinear equations

$$\frac{\partial u}{\partial t} - \mathrm{trace}(A(x)D^2 u) + H(x, Du) = 0, \quad (x,t) \in \mathbb{T}^N \times (0, +\infty), \tag{1.6}$$

one has first to establish *uniform* gradient bounds for the stationary equation

$$\epsilon \phi^\epsilon - \mathrm{trace}(A(x)D^2 \phi^\epsilon) + H(x, D\phi^\epsilon) = 0, \quad x \in \mathbb{T}^N, \quad \epsilon > 0. \tag{1.7}$$

By uniform gradient bounds, we mean

$$|D\phi^\epsilon|_\infty \le K, \quad \text{where } K \text{ is independent of } \epsilon. \tag{1.8}$$

This condition is crucial to be able to send ϵ to 0 in (1.7) in order to solve the ergodic problem (1.4), which is a first step when trying to prove (1.3).

Barles and Souganidis [7] obtained the first results concerning both estimates like (1.8) and the asymptotic behavior (1.3) for scalar equations ($m = 1$) with $A(x) = I$ in two contexts. The first one is for *sublinear Hamiltonians*, i.e., for Hamiltonians with a sublinear growth with respect to the gradient. A typical example is

$$H(x,p) = \langle b(x), p \rangle + \ell(x), \quad b \in C(\mathbb{T}^N; \mathbb{R}^N), \ \ell \in C(\mathbb{T}^N).$$

The second context is for *superlinear Hamiltonians*. The precise assumption is more involved (see (2.42)) and designed to allow the use of weak Bernstein-type arguments ([2]). The most important example is Hamiltonians with a superlinear growth with respect to the gradient

$$H(x,p) = a(x)|p|^{1+\alpha} + \ell(x), \quad \alpha > 0, \ a, \ell \in C(\mathbb{T}^N) \text{ and } a > 0. \tag{1.9}$$

As a consequence of these bounds together with the strong maximum principle, they obtain the convergence for the solutions of (1.6) when either H is sublinear or H is superlinear.

Using the extension of viscosity solutions to monotone systems of parabolic equations by Ishii and Koike [20], it is not difficult to adapt the results of [7] to the case of *sublinear* systems (1.2) (i.e., systems for which *all* the Hamiltonians are sublinear in the sense of [7]) on the one hand, and to the case of *superlinear* systems on the other hand. In this work, we focus on the more delicate issue of *asymmetric systems* like (1.5) containing both sublinear and superlinear Hamiltonians. The main difficulty is that the proofs of (1.8) in [7] are completely different in the two contexts. In the case of sublinear Hamiltonians, a method due to Ishii and Lions [21] allows to take profit of the uniform ellipticity of the equation to control the sublinear terms coming from the Hamiltonian. Whereas, up to our knowledge, the strategy to prove (1.8) for superlinear Hamiltonians relies on a weak Bernstein method needing an exponential-type change of function like $e^{w^\epsilon} = \phi^\epsilon - \min_{\mathbb{T}^N}\phi^\epsilon + 1$ in order to take advantage of the superlinear property of the Hamiltonian. In the case of systems, one has to perform this exponential change in all equations, producing quadratic terms of the form $|\sigma(x)Dw^\epsilon|^2$. These latter terms are dramatic in equations with sublinear Hamiltonians since they are not anymore under the control of the ellipticity using the usual proof.

The new idea to overcome this difficulty is to establish first uniform oscillation bounds

$$\mathrm{osc}(\phi^\epsilon) := \sup_{\mathbb{T}^N}\phi^\epsilon - \inf_{\mathbb{T}^N}\phi^\epsilon \le K \quad \text{where } K \text{ is independent of } \epsilon, \tag{1.10}$$

for the solution of (1.7). The point is that the proof of the oscillation bound does not need any exponential change of variable and so will work for asymmetric systems. This gives some uniform bounds for the new function w^ϵ and we are able to "localize" the proof (see (2.26) for details) allowing to control the bad quadratic term. It is worth mentioning that our proof of the oscillation bound works in very general settings and is a new result interesting by itself. For instance, (1.10) holds for the solutions of (1.7) as soon as

$$\frac{H(x,p)}{|p|} \to +\infty \quad \text{as } |p| \to +\infty \text{ uniformly with respect to } x. \tag{1.11}$$

Taking advantage of this oscillation bound, we are in fact able to produce a kind of unified proof of the sublinear and superlinear cases of [7]. More precisely, we obtain the gradient bound (1.8) for (1.7) when

$$H = \underline{H} + \overline{H},$$

where $\underline{H}$ is a sublinear Hamiltonian, i.e, having a sublinear growth,

$$|\overline{H}(x,p)| \le C(1+|p|),$$

and $\overline{H}$ is a superlinear one. An important feature, which will be crucial when dealing with asymmetric systems, is that we allow $\underline{H}$ or/and $\overline{H}$ to be zero which is natural for $\underline{H}$ but not so for $\overline{H}$. As far as the precise definition of superlinear Hamiltonian is concerned, we propose two definitions, one when (1.7) is uniformly elliptic, see (2.8), which generalizes slightly the one of [7] and a stronger one, see (2.40), which allows to deal with degenerate equations (1.7). Both of them include of course typical superlinear Hamiltonians like (1.9) and $\overline{H}$ may be zero in some cases. We refer the reader to Section 2.4 for comments and examples.

Using this new gradient bound for scalar equations, we are in position to extend it to systems

$$\begin{cases} \epsilon\phi_1^\epsilon - \mathrm{trace}(A_1(x)D^2\phi_1^\epsilon) + H_1(x, D\phi_1^\epsilon) + \phi_1^\epsilon - \phi_2^\epsilon = 0, \\ \epsilon\phi_2^\epsilon - \mathrm{trace}(A_2(x)D^2\phi_2^\epsilon) + H_2(x, D\phi_2^\epsilon) + \phi_2^\epsilon - \phi_1^\epsilon = 0, \quad x \in \mathbb{T}^N. \end{cases} \tag{1.12}$$

An immediate consequence is that we can solve the ergodic problem (1.4) extending the classical by now proofs of [24, 1] to the case of our systems. We then prove the convergence (1.3).

As in [7], the proof of the convergence is based on the strong maximum principle but let us mention that we establish a new version of the strong maximum principle for systems which may contain some degenerate equations, see Theorem 3.4 for details. In particular, the result holds for (1.5); See Section 3.5 for more examples and discussions.

Let us turn to an overview of related results in the litterature. The ideas of the proof of gradient bounds in viscosity theory using the uniform ellipticity of the equation are due to Ishii and Lions [21], see also [13, 3] and the references therein. Gradient bounds for superlinear-type Hamiltonians can be found in Lions [23] and Barles [2], see also Lions and Souganidis [25]. These ideas were used in Barles and Souganidis [7] as explained above. Our approach is mainly based on this latter work. For superlinear Hamiltonians satisfying $H(x,p) \geq a^{-1}|p|^m - a$, $a, m > 1$, some Hölder or gradient estimates were obtained in Capuzzo Dolcetta et al. [10], Barles [4], Cardaliaguet [11]. Recently, oscillations and Hölder bounds for nonlinear degenerate parabolic equations were proved in Cardaliaguet and Sylvestre [12] but the bounds depends on the L^∞ norm of the solution.

The large time behavior of such kind of nonlinear equations or systems in the periodic setting were extensively studied. For Hamilton-Jacobi equations (the totally degenerate case when $A \equiv 0$), we refer the reader to [28, 17, 6, 15, 5] and the references therein. In this framework, the gradient bounds are not a difficult step but the proof of the convergence is more delicate since one does not have any strong maximum principle. Such kind of results were extended to systems of Hamilton-Jacobi equations in [9, 27, 29, 26]. For second order nonlinear equations, the asymptotics results of [7] were recently generalized in [22] to some superlinear degenerate equations which are totally degenerate on some subset Σ of $\mathbb{T}^N$ and uniformly parabolic outside Σ using the gradient bound of Theorem 2.6 and some strong maximum principle type ideas. Similar results for uniformly convex degenerate equations were established in Cagnetti et al. [8] using a different approach based on a nonlinear adjoint method [16]. Their results also apply to systems with uniformly convex quadratic Hamiltonians with quite general degeneracy assumptions since the proof is not based on strong maximum principle-type arguments. However, such a method does not seem to be applicable for fully nonlinear equations and the system in [8] is not asymmetric.

The paper is organized as follows. The bounds (1.10)-(1.8) are established in Section 2 for the scalar equation (1.7) with $H = \underline{H} + \overline{H}$, firstly when the equation is uniformly elliptic and secondly in some degenerate cases. Some examples of applications are collected in Section 2.4. Section 3 is devoted to systems. The gradient bounds for asymmetric systems are obtained in Section 3.1 and a new strong maximum principle is obtained in Section 3.2. Then the ergodic problem is solved and the main application of large time behavior of asymmetric is investigated. The section ends with some examples of applications and extensions. Several results are collected in the appendix. In particular, since the equations under consideration do not satisfy the classical assumptions in viscosity solutions (due to the possibly superlinear growth of the Hamiltonian for instance), we recall several versions of the comparison principle which apply in our case. Finally a control theoritical interpretation is given.

Acknowledgement. We thank Guy Barles, Hiroyoshi Mitake and Hung Tran for fruitful discussions. This work was partially supported by the ANR (Agence Nationale de la Recherche) through HJnet project ANR-12-BS01-0008-01 and WKBHJ project ANR-12-BS01-0020.

2. Gradient bounds for nonlinear parabolic equations

For $\epsilon > 0$, we consider the approximated equation

$$\epsilon\phi^\epsilon - \mathrm{trace}(A(x)D^2\phi^\epsilon) + H(x, D\phi^\epsilon) = 0, \quad x \in \mathbb{T}^N. \tag{2.1}$$

We will always assume

$$A = \sigma\sigma^T, \ \sigma \in W^{1,\infty}(\mathbb{T}^N; \mathbb{R}^{N\times N}), \quad H \in C(\mathbb{T}^N \times \mathbb{R}^N). \tag{2.2}$$

2.1. **A general result for the oscillation.** We first show that the oscillation of the solution of (2.1) is uniformly bounded under a very general hypothesis. This result is interesting by itself.

$$\begin{cases} \text{There exists } L > 1 \text{ such that for all } x, y \in \mathbb{T}^N, \\ \text{if } |p| = L, \text{ then } H(x,p) \geq |p| \left[H(y, \dfrac{p}{|p|}) + |H(\cdot,0)|_\infty + N^{3/2}|\sigma_x|_\infty^2 \right]. \end{cases} \tag{2.3}$$

Notice that (2.3) is satisfied when (1.11) holds, see Section 2.4.

Lemma 2.1. *Assume* (2.2) *and* (2.3). *Let* ϕ^ϵ *be a continuous solution of* (2.1) *and let* $\phi^\epsilon(x_\epsilon) = \min \phi^\epsilon$. *Then*

$$\phi^\epsilon(x) - \phi^\epsilon(x_\epsilon) \leq L|x - x_\epsilon| \quad \textit{for all } x \in \mathbb{T}^N,$$

where L *is the constant (independent of* ϵ*) which appears in* (2.3).

An immediate consequence is

$$\mathrm{osc}(\phi^\epsilon) := \max \phi^\epsilon - \min \phi^\epsilon \leq \sqrt{N} L.$$

Proof of Lemma 2.1. For simplicity, we skip the ϵ superscript for ϕ^ϵ writing ϕ instead. The constant L which appears below is the one of (2.3). Consider

$$M = \max_{x,y\in\mathbb{T}^N} \{\phi(x) - L\phi(y) + (L-1)\min\phi - L|x-y|\}.$$

We are done if $M \leq 0$. Otherwise, the above positive maximum is achieved at $(\overline{x}, \overline{y})$ with $\overline{x} \neq \overline{y}$. Notice that the continuity of ϕ is crucial at this step. The theory of second order viscosity solutions (see [13] and Lemma 2.4) yields, for every $\varrho > 0$, the existence of $(p, X) \in \overline{J}^{2,+}\phi(\overline{x})$ and $(p/L, Y/L) \in \overline{J}^{2,-}\phi(\overline{y})$, $p = L\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|}$, such that

$$-\mathrm{trace}(A(\overline{x})X - A(\overline{y})Y) \geq -LN^{3/2}|\sigma_x|_\infty^2 + O(\varrho)$$

and

$$\begin{cases} \epsilon\phi(\overline{x}) - \mathrm{trace}(A(\overline{x})X) + H(\overline{x}, p) \leq 0, \\ \epsilon\phi(\overline{y}) - \mathrm{trace}(A(\overline{y})\dfrac{Y}{L}) + H(\overline{y}, \dfrac{p}{L})\} \geq 0. \end{cases}$$

It follows

$$\epsilon(\phi(\overline{x}) - L\phi(\overline{y})) - \mathrm{trace}(A(\overline{x})X - A(\overline{y})Y) + H(\overline{x}, p) - LH(\overline{y}, \frac{p}{L}) \leq 0.$$

We have

$$\epsilon(\phi(\overline{x}) - L\phi(\overline{y})) > -(L-1)\epsilon\min\phi \geq -L|H(\cdot,0)|_\infty$$

since $\epsilon \min \phi \leq |H(\cdot,0)|_\infty$ by the maximum principle (see (4.4)). Combining all the above information, we get

$$H(\overline{x},p) - L\left[H(\overline{y},\frac{p}{L}) + |H(\cdot,0)|_\infty + N^{3/2}|\sigma_x|_\infty^2\right] < 0.$$

Applying (2.3) yields a contradiction. $\square$

2.2. **Gradient bounds for uniformly elliptic equations.** In this section, we suppose that (2.1) is uniformly elliptic, i.e.,

$$\text{there exists } \nu > 0 \text{ such that } A(x) \geq \nu I, \quad x \in \mathbb{T}^N. \tag{2.4}$$

In this setting, we consider Hamiltonians under the special form $H = \underline{H} + \overline{H}$ where $\underline{H}$ is a sublinear Hamiltonian and $\overline{H}$ is of superlinear type in the sense defined below. This form will be useful later to deal with asymmetric systems. We rewrite (2.1) as

$$\epsilon\phi^\epsilon - \mathrm{trace}(A(x)D^2\phi^\epsilon) + \underline{H}(x,D\phi^\epsilon) + \overline{H}(x,D\phi^\epsilon) = 0. \tag{2.5}$$

We say that the $\underline{H}$ is a *sublinear Hamiltonian* if

$$|\underline{H}(x,p)| \leq \underline{C}(1+|p|), \ \ (x,p) \in \mathbb{T}^N \times \mathbb{R}^N. \tag{2.6}$$

We consider the following *superlinear*-type assumptions for $\overline{H}$. The first one is needed to obtain an oscillation bound for the solution and the second one is slightly stronger to get the gradient bound.

$$\begin{cases} \text{There exists } \overline{C} > 0, \ L, \mu > 1 \text{ such that} \\ \text{for all } x, y \in \mathbb{T}^N, \ |p| \geq L, \ \ \overline{H}(x,p) - \mu\overline{H}(y,\frac{p}{\mu}) \geq -\overline{C}|p|. \end{cases} \tag{2.7}$$

$$\begin{cases} \text{There exists } \overline{C} > 0, \ L > 1 \text{ such that} \\ \text{for all } x, y \in \mathbb{T}^N, \ |p| \geq L, \text{ and } \mu \geq 1 + L|x-y|, \\ \overline{H}(x,p) - \mu\overline{H}(y,\frac{p}{\mu}) \geq -\overline{C}|p|. \end{cases} \tag{2.8}$$

From the inequality in (2.6), we see that $\underline{H}$ has a sublinear growth in the classical sense. But, let us point out that the *superlinear* $\overline{H}$ may be zero in (2.7) and (2.8). This fact will allow to treat some cases of asymmetric systems. We chose to keep the terminology *superlinear* since (2.8) is a consequence of the superlinear-type assumption

$$\begin{cases} \overline{H} \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{T}^N \times \mathbb{R}^N), \text{ and there exists } L > 1 \text{ such that} \\ \text{for a.e. } x \in \mathbb{T}^N, \ |p| \geq L, \ L\left[\overline{H}_p p - \overline{H}\right] - |\overline{H}_x| \geq 0 \end{cases} \tag{2.9}$$

introduced in [7]. Moreover, (2.8) is satisfied for the typical superlinear Hamiltonian $\overline{H}(x,p) = a(x)|p|^{1+\alpha} + \ell(x)$, $\alpha > 0$, $a > 0$ we have in mind. We refer the reader to Section 2.4 for more discussions and examples showing that our assumptions are quite general.

We state the main result of this section.

Theorem 2.2. *Assume* (2.2), (2.4), (2.6) *and* (2.8). *For all* $\epsilon > 0$, *there exists a unique continuous viscosity solution* $\phi^\epsilon \in C(\mathbb{T}^N)$ *of* (2.5) *and a constant* $K > 0$ *independent of* ϵ *such that*

$$|D\phi^\epsilon|_\infty \leq K.$$

The proof relies on two important lemmas. The first one establishes a uniform bound for the oscillation and the second one improves this bound into a gradient bound. We first state and prove the lemmas and then give the proof of the theorem.

Lemma 2.3. *Under the hypotheses of Theorem 2.2, where* (2.8) *could be replaced by the weaker condition* (2.7)*, there exists a constant $K > 0$ (independent of ϵ) such that, if ϕ^ϵ is a continuous solution of* (2.5)*, then*

$$\operatorname{osc}(\phi^\epsilon) := \max \phi^\epsilon - \min \phi^\epsilon \le K.$$

Proof of Lemma 2.3. For simplicity, we skip the ϵ superscript for ϕ^ϵ writing ϕ instead.
1. Construction a concave test function. Consider the function

$$\Psi(s) = \frac{A_1}{A_2}(1 - \mathrm{e}^{-A_2 s}) \quad \text{for } 0 \le s \le \sqrt{N} = \operatorname{diameter}(\mathbb{T}^N), \tag{2.10}$$

where $A_1, A_2 > 0$ will be chosen later. It is straightforward to see that Ψ is a C^∞ concave increasing function satisfying $\Psi(0) = 0$ and, for all $s \in [0, \sqrt{N}]$,

$$\Psi'' + A_2\Psi' = 0, \quad A_1 e^{-A_2\sqrt{N}} = \Psi'(\sqrt{N}) \le \Psi'(s) \le \Psi'(0) = A_1. \tag{2.11}$$

2. Viscosity inequalities. Consider

$$M_\mu = \max_{x,y\in\mathbb{T}^N} \{\phi(x) + (\mu - 1)\min\phi - \mu\phi(y) - \Psi(|x - y|)\}, \tag{2.12}$$

with μ given in (2.7). If $M_\mu \le 0$ then the lemma holds with $K = A_1/A_2$. From now on, we argue by contradiction assuming that the maximum is positive and achieved at $(\overline{x}, \overline{y})$ with $\overline{x} \ne \overline{y}$.

The theory of second order viscosity solutions yields, for every $\varrho > 0$, the existence of $(p, X) \in \overline{J}^{2,+}\phi(\overline{x}), (p/\mu, Y/\mu) \in \overline{J}^{2,-}\phi(\overline{y})$ such that

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \varrho A^2, \tag{2.13}$$

with

$$q = \frac{\overline{x} - \overline{y}}{|\overline{x} - \overline{y}|}, \quad p = \Psi'(|\overline{x} - \overline{y}|)q, \quad B = \frac{1}{|\overline{x} - \overline{y}|}(I - q \otimes q), \tag{2.14}$$

$$A = \Psi'(|\overline{x} - \overline{y}|)\begin{pmatrix} B & -B \\ -B & B \end{pmatrix} + \Psi''(|\overline{x} - \overline{y}|)\begin{pmatrix} q\otimes q & -q\otimes q \\ -q\otimes q & q\otimes q \end{pmatrix} \tag{2.15}$$

and the following viscosity inequalities hold for the solution ϕ of (2.5),

$$\begin{cases} \epsilon\phi(\overline{x}) - \operatorname{trace}(A(\overline{x})X) + \underline{H}(\overline{x}, p) + \overline{H}(\overline{x}, p) \le 0, \\ \epsilon\phi(\overline{y}) - \operatorname{trace}(A(\overline{y})\frac{Y}{\mu}) + \underline{H}(\overline{y}, \frac{p}{\mu}) + \overline{H}(\overline{y}, \frac{p}{\mu}) \ge 0. \end{cases}$$

It follows

$$\begin{aligned} &\epsilon\phi(\overline{x}) - \epsilon\mu\phi(\overline{y}) - \operatorname{trace}(A(\overline{x})X - A(\overline{y})Y) \\ &\qquad + \underline{H}(\overline{x}, p) - \mu\underline{H}(\overline{y}, \frac{p}{\mu}) + \overline{H}(\overline{x}, p) - \mu\overline{H}(\overline{y}, \frac{p}{\mu}) \le 0. \end{aligned} \tag{2.16}$$

3. Trace estimates. We have the following estimates which will be useful in the sequel, see for instance [21, 7, 3]. A proof is given in the Appendix.

Lemma 2.4. *Under assumption* (2.2),

$$-\mathrm{trace}(A(\overline{x})X - A(\overline{y})Y) \geq -N|\sigma_x|_\infty^2 |\overline{x}-\overline{y}|\Psi'(|\overline{x}-\overline{y}|) + O(\varrho).$$

If, in addition, (2.4) *holds, then*

$$-\mathrm{trace}(A(\overline{x})X - A(\overline{y})Y) \geq -4\nu\Psi''(|\overline{x}-\overline{y}|) - \tilde{C}\Psi'(|\overline{x}-\overline{y}|) + O(\varrho), \tag{2.17}$$

where $\tilde{C} = \tilde{C}(N, \nu, |\sigma|_\infty, |\sigma_x|_\infty)$ *is given by* (4.3).

4. End of the proof. At $\hat{x}$ such that $\phi(\hat{x}) = \max\phi$, we get

$$\epsilon \min\phi + \underline{H}(\hat{x}, 0) + \overline{H}(\hat{x}, 0) \leq \epsilon\max\phi + \underline{H}(\hat{x}, 0) + \overline{H}(\hat{x}, 0) \leq 0.$$

It follows

$$\epsilon\min\phi \leq |(\underline{H} + \overline{H})(\cdot, 0)|_\infty =: \mathcal{H}_0. \tag{2.18}$$

So, using that the maximum is positive in (2.12), we obtain

$$\epsilon(\phi(\overline{x}) - \mu\phi(\overline{y})) > -(\mu-1)\epsilon\min\phi \geq -(\mu-1)\mathcal{H}_0.$$

We have

$$|\underline{H}(\overline{x}, p) - \mu\underline{H}(\overline{y}, \frac{p}{\mu})| \leq 2\underline{C}(1+|p|) = 2\underline{C}(1+\Psi'(|\overline{x}-\overline{y}|)),$$

from (2.6) and

$$-\mathrm{tr}(A(\overline{x})X - A(\overline{y})Y) \geq -4\nu\Psi''(|\overline{x}-\overline{y}|) - \tilde{C}\Psi'(|\overline{x}-\overline{y}|) + O(\varrho)$$

from Lemma 2.4 (2.17). Choosing

$$A_1 = Le^{A_2\sqrt{N}},$$

where L is the constant of (2.7), we obtain $|p| = \Psi'(|\overline{x}-\overline{y}|) \geq L$ from (2.11), (2.14) and

$$\overline{H}(\overline{x}, p) - \mu\overline{H}(\overline{y}, \frac{p}{\mu}) \geq -\overline{C}\Psi'(|\overline{x}-\overline{y}|)$$

from (2.7). Using these estimates in (2.16) and sending ϱ to 0, we have

$$-4\nu\Psi'' - (\tilde{C} + 2\underline{C} + \overline{C})\Psi' - (\mu-1)\mathcal{H}_0 - 2\underline{C} \leq 0. \tag{2.19}$$

Recalling that $\Psi'' + A_2\Psi' = 0$ by (2.11), we obtain a contradiction with (2.19) with the choice

$$A_2 = \frac{1}{4\nu}\left(\tilde{C} + 2\underline{C} + \overline{C} + \frac{(\mu-1)\mathcal{H}_0 + 2\underline{C} + 1}{L}\right).$$

It ends the proof. $\square$

Lemma 2.5. *Under the hypotheses of Theorem 2.2. Let* ϕ^ϵ *be a continuous viscosity solution of* (2.5) *and define*

$$\exp(w^\epsilon) = \phi^\epsilon - \min_{\mathbb{T}^N}\phi^\epsilon + 1.$$

Then, there exists a constant K *(independent of* ϵ*) such that* $|Dw^\epsilon|_\infty \leq K$.

Proof of Lemma 2.5. For simplicity, we skip the ϵ superscript.
1. New equation for w. The function w solves

$$\epsilon\, e^{-w(x)}(\min\phi - 1) + \epsilon - \mathrm{tr}(AD^2w) + \underline{G}(x,w,Dw) + \overline{G}(x,w,Dw) = 0, \tag{2.20}$$

where

$$\underline{G}(x,w,p) = e^{-w}\underline{H}(x,e^w p) - |\sigma(x)^T p|^2, \quad \overline{G}(x,w,p) = e^{-w}\overline{H}(x,e^w p). \tag{2.21}$$

2. Definition of the test function. We define $\Psi(s) = \frac{A_1}{A_2}(1-\mathrm{e}^{-A_2 s})$ as in (2.10) for $0 \le s \le \sqrt{N} = \mathrm{diameter}(\mathbb{T}^N)$ and set for further purpose

$$A_2 = \frac{1}{4\nu}\left(\tilde{C} + 4\underline{C} + \overline{C} + \mathcal{H}_0 + 2|\sigma|_\infty |\sigma_x|_\infty \mathrm{osc}(\phi)\right) \quad \text{and} \quad A_1 = (L + \mathrm{osc}(\phi))e^{A_2\sqrt{N}}, \tag{2.22}$$

where $\nu, \underline{C}, \overline{C}, L$ are the constants appearing in the assumptions (2.4), (2.6), (2.8), $\mathcal{H}_0$ is defined in (2.18) and $\mathrm{osc}(\phi)$ is bounded independently of ϵ by Lemma 2.3.

Since $\Psi(0) = 0$ and

$$\Psi(\sqrt{N}) = (L + \mathrm{osc}(\phi))\frac{e^{A_2\sqrt{N}} - 1}{A_2} \ge L + \mathrm{osc}(\phi) > \mathrm{osc}(\phi),$$

there exists $r \in [0, \sqrt{N}]$ such that

$$\Psi(r) = \mathrm{osc}(\phi). \tag{2.23}$$

We then consider

$$\max_{x,y \in \mathbb{T}^N} \{w(x) - w(y) - \Psi(|x-y|)\}. \tag{2.24}$$

If this maximum is nonpositive, then for all $x, y \in \mathbb{T}^N$, we have

$$w(x) - w(y) \le \Psi(|x-y|) \le A_1|x-y|,$$

where the latter inequality follows from the concavity of Ψ. This yields the desired result.

From now on, we argue by contradiction, assuming that the maximum in (2.24) is positive. This implies that it is achieved at $(\overline{x}, \overline{y}) \in \mathbb{T}^N \times \mathbb{T}^N$ with $\overline{x} \neq \overline{y}$ since w is continuous. Noticing that

$$w(\overline{x}) - w(\overline{y}) \le e^{w(\overline{x}) - w(\overline{y})} - 1 \le e^{w(\overline{x})} - 1 \le \phi(\overline{x}) - \min\,\phi \le \mathrm{osc}(\phi), \tag{2.25}$$

we get

$$0 < w(\overline{x}) - w(\overline{y}) - \Psi(|\overline{x} - \overline{y}|) \le \mathrm{osc}(\phi) - \Psi(|\overline{x} - \overline{y}|).$$

Using that Ψ is increasing and (2.23), we infer

$$|\overline{x} - \overline{y}| < r. \tag{2.26}$$

This latter inequality is a kind of localization of points of maxima in (2.24). It will allow us to control the quadratic term coming from (2.21) in term of the oscillation, see (2.36).
3. Viscosity inequalities for (2.20). Writing the viscosity inequalities as in Step 2 of the proof of Lemma 2.3, we obtain

$$\epsilon e^{-w(\overline{x})}(\min\phi - 1) + \epsilon - \mathrm{trace}(A(\overline{x})X) + \underline{G}(\overline{x}, w(\overline{x}), p) + \overline{G}(\overline{x}, w(\overline{x}), p) \le 0,$$
$$\epsilon e^{-w(\overline{y})}(\min\phi - 1) + \epsilon - \mathrm{trace}(A(\overline{y})Y) + \underline{G}(\overline{y}, w(\overline{y}), p) + \overline{G}(\overline{y}, w(\overline{y}), p) \ge 0.$$

Therefore,

$$\begin{aligned}(2.27)\qquad &\epsilon(e^{-w(\overline{x})}-e^{-w(\overline{y})})(\min\phi-1)-\mathrm{trace}(A(\overline{x})X-A(\overline{y})Y)\\ &+\underline{G}(\overline{x},w(\overline{x}),p)-\underline{G}(\overline{y},w(\overline{y}),p)+\overline{G}(\overline{x},w(\overline{x}),p)-\overline{G}(\overline{y},w(\overline{y}),p)\leq 0.\end{aligned}$$

The end of the proof consists in reaching a contradiction in the above inequality.
4. Estimates of the terms in (2.27). From (2.18) and the fact that $w(\overline{x})>w(\overline{y})\geq 0$, we have

$$\epsilon(e^{-w(\overline{x})}-e^{-w(\overline{y})})(\min\phi-1)>-\mathcal{H}_0. \tag{2.28}$$

From Lemma 2.4 (2.17), we have

$$-\mathrm{trace}(A(\overline{x})X-A(\overline{y})Y)\geq -4\nu\Psi''(|\overline{x}-\overline{y}|)-\tilde{C}\Psi'(|\overline{x}-\overline{y}|)+O(\varrho). \tag{2.29}$$

Using (2.2), (2.6) and recalling that $|p|=\Psi'(|\overline{x}-\overline{y}|)$, we get

$$\begin{aligned}(2.30)\qquad &|\underline{G}(\overline{x},w(\overline{x}),p)-\underline{G}(\overline{y},w(\overline{y}),p)|\\ \leq\ & |e^{-w(\overline{x})}\underline{H}(\overline{x},e^{w(\overline{x})}p)|+|e^{-w(\overline{y})}\underline{H}(\overline{y},e^{w(\overline{y})}p)|+||\sigma(\overline{x})^Tp|^2-|\sigma(\overline{y})^Tp|^2|\\ \leq\ & 2\underline{C}(1+\Psi'(|\overline{x}-\overline{y}|))+2|\sigma|_\infty|\sigma_x|_\infty|\overline{x}-\overline{y}|\Psi'^2(|\overline{x}-\overline{y}|).\end{aligned}$$

We now estimate $\overline{G}(\overline{x},w(\overline{x}),p)-\overline{G}(\overline{y},w(\overline{y}),p)$ using (2.8). Set $P:=e^{w(\overline{x})}p$ and $\mu:=e^{w(\overline{x})-w(\overline{y})}$, we have

$$\overline{G}(\overline{x},w(\overline{x}),p)-\overline{G}(\overline{y},w(\overline{y}),p)\ =\ e^{-w(\overline{x})}\left(\overline{H}(\overline{x},P)-\mu\overline{H}(\overline{y},\frac{P}{\mu})\right). \tag{2.31}$$

From the choice of A_1 in (2.22) and the concavity of Ψ, we get

$$|P|\geq|p|=\Psi'(|\overline{x}-\overline{y}|)\geq\Psi'(r)=A_1e^{-A_2r}=(L+\mathrm{osc}(\phi))e^{A_2(\sqrt{N}-r)}\geq L. \tag{2.32}$$

Since the maximum in (2.24) is positive, it follows

$$\mu\geq 1+w(\overline{x})-w(\overline{y})>1+\Psi(|\overline{x}-\overline{y}|)\geq 1+\Psi'(|\overline{x}-\overline{y}|)|\overline{x}-\overline{y}|\geq 1+L|\overline{x}-\overline{y}|. \tag{2.33}$$

From (2.25) and since $|p|\leq\Psi'(0)=A_1$, we notice

$$L\leq|p|\leq A_1=A_1(\sigma,\underline{C},\overline{C},L,\mathrm{osc}(\phi))\quad\text{and}\quad 1+L|\overline{x}-\overline{y}|\leq\mu\leq\mathrm{osc}(\phi)+1, \tag{2.34}$$

which will be useful in the proof of Theorem 2.2. It follows that we can apply (2.8) to (2.31) to get

$$\overline{G}(\overline{x},w(\overline{x}),p)-\overline{G}(\overline{y},w(\overline{y}),p)\geq-\overline{C}e^{-w(\overline{x})}|P|=-\overline{C}\Psi'(|\overline{x}-\overline{y}|). \tag{2.35}$$

Plugging (2.28), (2.29), (2.30) and (2.35) in (2.27) and by letting $\varrho\to 0$, we obtain

$$-4\nu\Psi''(|\overline{x}-\overline{y}|)-(\tilde{C}+2\underline{C}+\overline{C})\Psi'(|\overline{x}-\overline{y}|)-2|\sigma|_\infty|\sigma_x|_\infty|\overline{x}-\overline{y}|\Psi'^2(|\overline{x}-\overline{y}|)-\mathcal{H}_0-2\underline{C}<0.$$

Since $|\overline{x}-\overline{y}|\leq r$ by (2.26), using that for all $s\in[0,r]$, $\Psi''(s)+A_2\Psi'(s)=0$ and

$$|\overline{x}-\overline{y}|\Psi'^2(|\overline{x}-\overline{y}|)\leq\Psi(|\overline{x}-\overline{y}|)\Psi'(|\overline{x}-\overline{y}|)\leq\Psi(r)\Psi'(|\overline{x}-\overline{y}|)=\mathrm{osc}(\phi)\Psi'(|\overline{x}-\overline{y}|), \tag{2.36}$$

we can rewrite the above estimate as

$$\left(4\nu A_2-(\tilde{C}+2\underline{C}+\overline{C}+2|\sigma|_\infty|\sigma_x|_\infty\mathrm{osc}(\phi))\right)\Psi'(|\overline{x}-\overline{y}|)-\mathcal{H}_0-2\underline{C}<0.$$

It is then straightforward to see that (2.22) leads to a contradiction in the above inequality. Finally, we obtain the result with $K=A_1$. □

We turn to the proof of Theorem 2.2. To use the previous lemmas, we need first to build a *continuous* viscosity solution of (2.5). It is straightforward to build a discontinuous viscosity solution for (2.5) using Perron's method for viscosity solutions of second order equations, see [19, 13]. The continuity of this solution usually follows from a strong comparison principle for (2.5), i.e., a comparison principle between USC viscosity subsolutions and LSC viscosity supersolutions. But, classical structure assumptions on the first order nonlinearity $\underline{H}+\overline{H}$ like Lipschitz continuity in x-variable, do not hold here. It follows that a strong comparison result may not hold. We are only able to compare a discontinuous sub- or supersolution with a Lipschitz continuous solution, see Theorem 4.1 in the Appendix. That is why, we need another approach inspired from [7] to build a continuous solution of (2.5). It is based on a truncation of the nonlinearity. Another natural approach would be to use the classical regularity theory for uniformly elliptic equations but it would not apply for degenerate equations we will consider in the next section.

Proof of Theorem 2.2. We fix $\epsilon > 0$ and skip the ϵ-dependence in the proof for simplicity.
1. Truncated Hamiltonian. For all $n > 0$, we define the continuous Hamiltonian

$$H_n := \underline{H}_n + \overline{H}_n \tag{2.37}$$

with

$$\underline{H}_n(x,p) \text{ (resp. } \overline{H}_n(x,p)) = \begin{cases} \underline{H}(x,p) \text{ (resp. } \overline{H}(x,p)) & \text{if } |p| \leq n, \\ \underline{H}(x,n\frac{p}{|p|}) \text{ (resp. } \overline{H}(x,n\frac{p}{|p|})) & \text{if } |p| \geq n. \end{cases}$$

2. Construction of a continuous viscosity solution for the truncated equation. We have a strong comparison principle between discontinuous solutions for (2.5) where $\underline{H},\overline{H}$ are replaced with $\underline{H}_n,\overline{H}_n$ respectively, see Theorem 4.2. By Perron's method, see [13, 19], there exists a continuous viscosity solution ϕ_n.
3. Uniform Lipschitz bound for solutions of the truncated equation. We first notice that $\underline{H}_n$ satisfies (2.6) with the same constant $\underline{C}$ as $\underline{H}$. Moreover, if we choose n bigger than L which appears in (2.7), then, by Lemma 2.3, we obtain a bound for the oscillation of ϕ_n which is independent of n,ϵ. Moreover, if n is chosen bigger than the right hand side of (2.34), then (2.8) hold for $\overline{H}_n$ with the same constants as for $\overline{H}$ for all p satisfying (2.34). As noticed in the proof of Lemma 2.5, it is enough to obtain a gradient bound K for w_n defined by

$$\exp(w_n) = \phi_n - \min_{\mathbb{T}^N} \phi_n + 1.$$

The crucial point is that this gradient bound K does not depend on n since the constants in (2.6), (2.7), (2.8) are the same for all $\underline{H}_n,\overline{H}_n$. The uniform bound for the oscillation of ϕ_n yields a L^∞ bound for w_n which is independent of n,ϵ. It follows

$$|D\phi_n|_\infty \leq K\exp(2\sqrt{N}K). \tag{2.38}$$

4. Convergence of ϕ_n. In addition to (2.38), from (4.4), we have a L^∞ bound for ϕ_n which is independent of n. Then, by Ascoli-Arzela's theorem, we obtain that, up to a subsequence, $\phi_n \to \phi$ in $C(\mathbb{T}^N)$ and $\phi = \phi^\epsilon$ is Lipschitz continuous with $|D\phi|_\infty \leq Ke^{2\sqrt{N}K}$, which is independent of ϵ. Noticing that $\underline{H}_n \to \underline{H}, \overline{H}_n \to \underline{H}$. By the stability result for viscosity solution, we conclude that ϕ is a Lipschitz continuous viscosity solution of (2.5).
5. Uniqueness. The uniqueness of ϕ in the class of continuous viscosity solutions relies on a comparison principle between the Lipschitz solution ϕ with any continuous solution $\tilde{\phi}$ of (2.5), see Theorem 4.1 in the Appendix. □

2.3. **Gradient bounds for degenerate elliptic equations.** We now consider degenerate elliptic equations, i.e., (2.4) does not necessarily hold. In this case, we suppose that the sublinear Hamiltonian $\underline{H} \equiv 0$ in (2.5) and we reinforce the assumptions (2.7)-(2.8) in order that the superlinear Hamiltonian $\overline{H}$ controls all the terms in the equation.

The assumptions are

$$\left\{ \begin{array}{l} \text{There exists } L, \mu > 1 \text{ such that} \\ \text{for all } x, y \in \mathbb{T}^N,\ |p| \geq L, \\ \overline{H}(x,p) - \mu \overline{H}(y, \frac{p}{\mu}) \geq (\mu - 1)\left(|\overline{H}(\cdot, 0)|_\infty + N|\sigma_x|^2_\infty |p|\right), \end{array} \right. \tag{2.39}$$

$$\left\{ \begin{array}{l} \text{There exists } L > 1 \text{ such that} \\ \text{for all } x, y \in \mathbb{T}^N,\ |p| \geq L \text{ and } \mu \geq 1 + L|x-y|, \\ \overline{H}(x,p) - \mu \overline{H}(y, \frac{p}{\mu}) \geq (\mu - 1)\left(|\overline{H}(\cdot, 0)|_\infty + N|\sigma_x|^2_\infty |p| + 2|\sigma_x|_\infty |\sigma|_\infty |p|\right). \end{array} \right. \tag{2.40}$$

As for the uniformly elliptic case, the first assumption is needed to get the oscillation bound and the stronger one to obtain the gradient bound. Some discussion about these assumptions and examples are given in Section 2.4.

Theorem 2.6. *Assume* (2.2), (2.40) *and suppose that* $\underline{H} \equiv 0$. *For all* $\epsilon > 0$, *there exists a unique continuous viscosity solution* $\phi^\epsilon \in C(\mathbb{T}^N)$ *of* (2.5) *and a constant* $K > 0$ *independent of* ϵ *such that*

$$|D\phi^\epsilon|_\infty \leq K.$$

The proof of the above theorem is similar to the one of Theorem 2.2, so we skip it. It relies on the auxiliary Lemmas 2.3 and 2.5 where (2.7)-(2.8) are replaced by (2.39)-(2.40). The proofs of the auxiliary lemmas follow the same lines, the changes occur in the estimates of the terms in (2.16) and (2.27), so we only rewrite Step 4 of the proof of Lemma 2.5 where the main changes occur.

Proof of Lemma 2.5 for Theorem 2.6 under assumptions (2.40). We estimate the different terms appearing in (2.27). We set $P := e^{w(\overline{x})}p$ and $\mu := e^{w(\overline{x})-w(\overline{y})}$ and we recall that, if the maximum is positive in (2.24) and with a suitable choice of A_1, A_2, r in (2.10), then

$$|P| = |e^{w(\overline{x})}p| \geq L > 1 \text{ and } \mu > 1 + |p||\overline{x} - \overline{y}| \geq 1 + L|\overline{x} - \overline{y}|$$

(see (2.32) and (2.33)). From (2.18), we get

$$\begin{array}{rcl} \epsilon(e^{-w(\overline{x})} - e^{-w(\overline{y})})(\min \phi - 1) & > & e^{-w(\overline{x})}(1-\mu)\epsilon(\min \phi - 1) \\ & \geq & -e^{-w(\overline{x})}(\mu - 1)|\overline{H}(\cdot, 0)|_\infty. \end{array} \tag{2.41}$$

From Lemma 2.4 (2.17), we have

$$-\mathrm{trace}(A(\overline{x})X - A(\overline{y})Y) \geq -e^{-w(\overline{x})}(\mu - 1)N|\sigma_x|^2_\infty |P| + O(\varrho).$$

Since $\underline{H} \equiv 0$, we have

$$\begin{array}{rcl} \underline{G}(\overline{x}, w(\overline{x}), p) - \underline{G}(\overline{y}, w(\overline{y}), p) & = & -|\sigma(\overline{x})^T p|^2 + |\sigma(\overline{y})^T p|^2 \\ & \geq & -2|\sigma|_\infty |\sigma_x|_\infty |\overline{x} - \overline{y}||p|^2 \\ & \geq & -2e^{-w(\overline{x})}(\mu - 1)|\sigma|_\infty |\sigma_x|_\infty |P|. \end{array}$$

As far as the superlinear Hamiltonians are concerned, we have

$$\overline{G}(\overline{x}, w(\overline{x}), p) - \overline{G}(\overline{y}, w(\overline{y}), p) \;=\; e^{-w(\overline{x})}\left(\overline{H}(\overline{x}, P) - \mu\overline{H}(\overline{y}, \frac{P}{\mu})\right).$$

Plugging the previous estimates in (2.27) and applying (2.40), we reach a contradiction with the strict inequality in (2.41). It ends the proof. □

2.4. **Comments on the assumptions, examples and extensions.**

Example 2.7. (Hamiltonian satisfying (2.3)) If $\limsup_{|p|\to+\infty} \frac{H(x,p)}{|p|} = +\infty$ uniformly with respect to x then (2.3) holds. For instance, if $H(x,p) = a(x)\sin(h(|p|))|p|^{1+\alpha} + \ell(x)$, where $\alpha > 0$, $a > 0, \ell, h$ are continuous and $\limsup_{r\to+\infty} \sin(h(r)) > 0$, then (2.3) holds.

Lemma 2.8.

(i) *A sublinear Hamiltonian* $\underline{H}$ *satisfying* (2.6) *satisfies* (2.7).

(ii) *If there exists* $\alpha \geq 0$, $a > 0$, $A, B, L \geq 0$ *such that, for all* $x \in \mathbb{T}^N$, $|p| \geq L$,

$$A|p|^\alpha + B|p| \geq H(x,p) \geq a|p|^\alpha - B|p|,$$

then (2.7) *holds*

Proof of Lemma 2.8. (i) Using (2.6), we have, for $\mu = 2$ and $|p| > 1$,

$$\underline{H}(x,p) - \mu\underline{H}(y, \frac{p}{\mu}) \geq -C(1+|p|) - C\mu(1+|\frac{p}{\mu}|) \geq -3C(1+|p|) \geq -6C|p|.$$

(ii) When $0 \leq \alpha \leq 1$, then (2.6) holds. For $\alpha > 1$,

$$H(x,p) - \mu H(y, \frac{p}{\mu}) \geq (a - \mu^{1-\alpha}A)|p|^\alpha - 2B|p| \geq -2B|p|$$

provided $\mu \geq (A/a)^{1/(\alpha-1)}$. □

Notice that no regularity assumption (except continuity) is needed to obtain the oscillation bound. To obtain a gradient bound, we need to reinforce (2.7)-(2.39) into (2.8)-(2.40). These new assumptions contain a kind of regularity assumption with respect to (x,p). Actually, (2.8) is very close to (2.9) in [7] and (2.40) is very close to

$$\text{(2.42)} \qquad \begin{cases} \overline{H} \in W^{1,\infty}_{\text{loc}}(\mathbb{T}^N \times \mathbb{R}^N) \text{ and there exists } L \text{ such that} \\ \text{if } |p| \geq L, \text{ then, for a.e. } (x,p) \in \mathbb{T}^N \times \mathbb{R}^N, \\ L\left[(\overline{H})_p p - \overline{H} - |\overline{H}(\cdot,0)|_\infty - N|\sigma_x|^2_\infty |p| - 2|\sigma|_\infty |\sigma_x|_\infty |p|\right] - |(\overline{H})_x| \geq 0, \end{cases}$$

which is an extension of (2.9) for x-dependent degenerate diffusion matrices. In (2.9)-(2.42), the Hamiltonian is supposed to be locally Lipschitz with respect to (x,p). When the Hamiltonians are locally Lipschitz, we can prove that (2.8)-(2.40) and (2.9)-(2.42) are equivalent but our assumptions allow to deal with some non Lipschitz continuous Hamiltonians as shown in the following example.

Example 2.9. (A non Lipschitz continuous Hamiltonian satisfying (2.8)) Let $H(x,p) = |p|^2 + h(x,p)$ with h continuous bounded. For all $x, y \in \mathbb{T}^N, p \in \mathbb{R}^N$ and $\mu > 1$, we have

$$H(x,p) - \mu H(y, \frac{p}{\mu}) = (1 - \frac{1}{\mu})|p|^2 + h(x,p) - \mu h(y, \frac{p}{\mu}) \geq -(1+\mu)|h|_\infty.$$

From (2.34), we see that (2.8) has to hold only for bounded $\mu \geq 1 + L|x-y|$ and $|p| \geq L > 1$. So (2.8) holds.

We turn to some examples of sub- and superlinear Hamiltonians.

Example 2.10. (Typical sublinear Hamiltonians from optimal control problem)

$$\underline{H}(x,p) = \sup_{\theta\in\Theta}\{-\langle b_\theta(x), p\rangle - \ell_\theta(x)\},$$

where $b_\theta, \ell_\theta \in W^{1,\infty}(\mathbb{T}^N)$ uniformly with respect to θ. Such a $\underline{H}$ satisfies (2.6). Notice that the Lipschitz continuity is actually not needed in the proof of the gradient bound.

Example 2.11. (Typical superlinear Hamiltonian)

$$\overline{H}(x,p) = a(x)|p|^{1+\alpha} + b(x)|p| + c(x),$$

with $a > 0, b, c \in W^{1,\infty}(\mathbb{T}^N)$. Then $\overline{H}$ satisfies (2.8) and (2.40) as soon as (2.2) holds. It follows that the gradient bound of Theorem 2.2 holds for (2.5) even if σ is degenerate (i.e., (2.4) does not hold).

Example 2.12. $H(x,p) = |B(x)p|^k + \langle b(x), p\rangle + \ell(x)$ with $B \in C(\mathbb{T}^N; \mathbb{R}^{N\times N})$, $b \in C(\mathbb{T}^N; \mathbb{R}^N)$, $\ell \in C(\mathbb{T}^N)$ satisfies (2.8) if

$$|b_x|_\infty, |B_x|_\infty \le C \quad \text{and} \quad B(x)B(x)^T > 0.$$

Example 2.13. Let define $\hat{H}(p) = \hat{H}(|p|)$ radial by $\hat{H}(0) = 0$ and, for all $t \in [n, n+1]$, $\hat{H}(t) = (n+1)t - n(n+1)/2$. We notice that $\hat{H} \in W^{1,\infty}_{\rm loc}(\mathbb{R}^N)$ and $\hat{H}_p(p)p - \hat{H} = 0$ a.e. It follows that $H(x,p) = \hat{H}(p) + \ell(x)$ satisfies (2.8) if ℓ is continuous but does not satisfy (2.40) (even if ℓ is Lipschitz continuous). The Hamiltonian $H(x,p) = \underline{H}(x,p) + \hat{H}(p) + \ell(x)$ where $\underline{H}$ satisfies (2.6) fulfills the assumptions of Theorem 2.2 if the diffusion matrix satisfies (2.2)-(2.4) but such a case cannot be handled with the results of [7].

Remark 2.14. We can extend the results with easy adaptations to fully nonlinear equations like

$$\epsilon\phi^\epsilon + \sup_{\theta\in\Theta}\{-\mathrm{trace}(A_\theta(x)D^2\phi^\epsilon) + \underline{H}_\theta(x, D\phi^\epsilon) + \overline{H}_\theta(x, D\phi^\epsilon)\} = 0, \quad x \in \mathbb{T}^N, \tag{2.43}$$

where Θ is a compact metric space and there exists a constant $C > 0$ such that $A_\theta = \sigma_\theta\sigma_\theta^T$ satisfies

$$|\sigma_\theta(x)| \le C, \quad |\sigma_\theta(x) - \sigma_\theta(y)| \le C|x-y|, \qquad x, y \in \mathbb{T}^N, \theta \in \Theta,$$

and both $\underline{H}_\theta, \overline{H}_\theta$ are continuous satisfying, for all $R > 0$, there exists a modulus of continuity m_R such that

$$|H_\theta(x,0)| \le C, \quad |H_\theta(x,p) - H_\theta(y,p)| \le m_R(|x-y|), \quad x, y \in \mathbb{T}^N, |p| \le R, \theta \in \Theta. \tag{2.44}$$

Then

- Theorem 2.2 holds when A_θ satisfies (2.4), $\underline{H}_\theta$ satisfies (2.6) and $\overline{H}_\theta$ satisfies (2.8), uniformly with respect to θ.
- Theorem 2.6 holds in particular when $\underline{H}_\theta \equiv 0$ and $\overline{H}_\theta$ satisfies (2.40) uniformly with respect to θ.

3. Asymmetric Systems

We consider the weakly coupled system

$$(3.1)\qquad \begin{cases} e\phi_1^\epsilon - \mathrm{trace}(A_1(x)D^2\phi_1) + H_1(x, D\phi_1) + \phi_1 - \phi_2 = 0, \ x \in \mathbb{T}^N, \\ e\phi_2^\epsilon - \mathrm{trace}(A_2(x)D^2\phi_2) + H_2(x, D\phi_2) + \phi_2 - \phi_1 = 0, \end{cases}$$

where the A_i's and H_i's satisfy the steady assumption (2.2).

3.1. **Gradient bounds for systems.** The aim is to obtain uniform gradient bounds (i.e., independent of ϵ) for (3.1) when $H_i = \underline{H}_i + \overline{H}_i$. We define

$$(3.2)\qquad \mathcal{H} := \sup_{x\in\mathbb{T}^N, 1\le j\le 2} |H_j(x, 0)|.$$

and each equation satisfies a different set of assumptions.

$$(3.3)\qquad A_1, \underline{H}_1, \overline{H}_1 \text{ satisfy } (2.4),(2.6),(2.8) \text{ respectively},$$

$$(3.4)\qquad \underline{H}_2 \equiv 0 \text{ and } \overline{H}_2 \text{ satisfies (2.40) with } |\overline{H}(\cdot, 0)| \text{ replaced by } 3\mathcal{H}.$$

Assumption (3.3) means that the first equation is of uniformly elliptic type with sublinear Hamiltonian whereas (3.4) tells that the second one may be degenerate with superlinear Hamiltonian. So the system is asymmetric. This case is the one in interest in this work but let us mention that the case when both equations satisfy either (3.3) or (3.4) is also possible with easier arguments in the proof of the theorem which follows. See Section 3.5 for examples and extensions.

Theorem 3.1. *Assume* (3.3)-(3.4)*. There exists a unique continuous viscosity solution* $\phi^\epsilon = (\phi_1^\epsilon, \phi_2^\epsilon) \in C(\mathbb{T}^N)^2$ *of* (3.1) *and* $K > 0$ *depending only on the* H_i*'s such that*

$$|D\phi_i^\epsilon|_\infty \le K \quad \textit{for all } \epsilon > 0,\ i = 1, 2.$$

Similarly to the case of scalar equations, the proof consists in two main steps: first, we prove an uniform bound for the oscillation and we then improve it to a uniform gradient bound. The key lemmas are

Lemma 3.2. *Under the hypotheses of Theorem 3.1. Let* $\phi^\epsilon = (\phi_1^\epsilon, \phi_2^\epsilon) \in C(\mathbb{T}^N)^2$ *be a solution of* (3.1)*. There exists* $K > 0$ *depending only on the* H_i*'s such that*

$$\mathrm{osc}(\phi_i^\epsilon) \le K \quad \textit{for all } \epsilon > 0,\ i = 1, 2.$$

We skip the proof of Lemma 3.2 since it is similar to the case of scalar equations.

Lemma 3.3. *Under the hypotheses of Theorem 3.1, let* ϕ^ϵ *be a continuous viscosity solution of* (3.1) *and define* w^ϵ *by*

$$\exp(w_i^\epsilon) = \phi_i^\epsilon - \min_{\mathbb{T}^N} \phi_i^\epsilon + 1 \quad \textit{for all } \epsilon,\ i.$$

Then, there exists a constant K *independent of* ϵ *such that* $|Dw_i^\epsilon|_\infty \le K$ *for all* ϵ, i.

Proof of Theorem 3.1. It is sufficient to see that the proof of Theorem 2.2 can be extended to systems like (3.1). For $i = 1, 2$, we define H_{in} like in (2.37). By Perron's method for systems, see [20] and Theorem 4.2, there exists a continuous viscosity solution ϕ_n of (3.1) where the H_i are replaced with the H_{in}'s. It is now possible to apply Lemma 3.3 to ϕ_n to obtain a gradient bound which is independent of ϵ, n. We conclude as in the proof of Theorem 2.2. □

Proof of Lemma 3.3. The proof is almost the same as the one for scalar equations. The main change is the estimate (3.7). So we only write some important steps.
1. Set $m_i = \min_{\mathbb{T}^N} \phi_i$, the new equation satisfied by w_i is

$$-\mathrm{trace}(A_i(x)D^2 w_i) + \underline{G}_i(x, w_i, Dw_i) + \overline{G}_i(x, w_i, Dw_i) + b_i(w_i) + 1 - e^{w_{i+1}-w_i} = 0,$$

where

$$\underline{G}_i(x, w, p) = e^{-w}\underline{H}_i(x, e^w p) - |\sigma_i(x)^T p|^2, \quad \overline{G}_i(x, w, p) = e^{-w}\overline{H}_i(x, e^w p),$$
$$b_i(w) = e^{-w}[m_i - m_{i+1} + \epsilon(m_i - 1)] + \epsilon, \quad \text{where we identify } m_3 = m_1 \text{ and } w_3 = w_1.$$

Consider

$$\max_{\mathbb{T}^N \times \mathbb{T}^N,\, i=1,2} \{w_i(x) - w_i(y) - \Psi(|x-y|)\}, \tag{3.5}$$

where Ψ is of the form in (2.10) with A_1, A_2 to be chosen later. We are done if the maximum is nonpositive.
2. Otherwise, the maximum is positive and hence is achieved at $(\overline{x}, \overline{y})$ with $\overline{x} \neq \overline{y}$ for some $i \in \{1, 2\}$. So we can write the viscosity inequalities for the ith equation at this point. For every $\varrho > 0$, there exist $(p, X) \in \overline{J}^{2,+} w_i(\overline{x})$, and $(p, Y) \in \overline{J}^{2,-} w_i(\overline{y})$ such that (2.13)-(2.14)-(2.15) hold and (2.27) is replaced by

$$\underline{\mathcal{G}}_i + \overline{\mathcal{G}}_i + \mathcal{B} + \mathcal{C} \leq 0, \tag{3.6}$$

where

$$\begin{aligned}
\underline{\mathcal{G}}_i &= -\mathrm{tr}(A_i(\overline{x})X - A_i(\overline{y})Y) + \underline{G}_i(\overline{x}, w_i(\overline{x}), p) - \underline{G}_i(\overline{y}, w_i(\overline{y}), p),\\
\overline{\mathcal{G}}_i &= \overline{G}_i(\overline{x}, w_i(\overline{x}), p) - \overline{G}_i(\overline{y}, w_i(\overline{y}), p),\\
\mathcal{B} &= b_i(w_i(\overline{x})) - b_i(w_i(\overline{y})),\\
\mathcal{C} &= e^{w_{i+1}(\overline{y}) - w_i(\overline{y})} - e^{w_{i+1}(\overline{x}) - w_i(\overline{x})}, \quad \text{where we identify } w_3 = w_1.
\end{aligned}$$

Since the maximum in (3.5) is positive, we have $\mathcal{C} \geq 0$. Next, we claim that

$$m_i - m_{i+1} + \epsilon m_i \leq 3\mathcal{H}, \tag{3.7}$$

see Step 5 for its proof. It follows easily that

$$\mathcal{B} \geq -3\mathcal{H}.$$

Since we have different assumptions for each equation, we distinguish two cases: "uniformly elliptic" (Step 3) and "degenerate elliptic" (step 4) to get a contradiction in (3.6).
3. When $i = 1$, since the first equation is "uniformly elliptic", we use the arguments of Step 4 in the proof of Lemma 2.5. We obtain

$$\begin{aligned}
&\underline{\mathcal{G}}_1 + \overline{\mathcal{G}}_1 + \mathcal{B} + \mathcal{C} \\
\geq\ & -4\nu\Psi'' - (\tilde{C} + 2\underline{C} + \overline{C} + 2|\sigma_1|_\infty |(\sigma_1)_x|_\infty |\overline{x} - \overline{y}|\Psi')\Psi' - 2\underline{C} - 3\mathcal{H}.
\end{aligned} \tag{3.8}$$

With a suitable choice of A_1, A_2 in the definition of Ψ, there exists $r \geq |\overline{x} - \overline{y}|$ such that $|\overline{x} - \overline{y}|\Psi' \leq \Psi(r) = \max_j \mathrm{osc}(\phi_j)$, which is bounded by Lemma 3.2. It is then possible to make the right-hand side of (3.8) positive.
4. When $i = 2$, since the second equation is "degenerate elliptic" so we control the 0th order terms using the superlinear Hamiltonian. We repeat readily the arguments of the proof of Lemma 2.5 in Section 2.3 (case of degenerate elliptic equations) estimating $\overline{\mathcal{G}}_2$ using now (3.4). We omit the details.

Finally Steps 3 and 4 lead to a contradiction in (3.6).
5. It remains to prove (3.7). Let x_1 such that $\phi_1(x_1) = \min_{\mathbb{T}^N} \phi_1$. Using the first equation of (3.1), we get

$$\min_{\mathbb{T}^N} \phi_2 - \min_{\mathbb{T}^N} \phi_1 \le \phi_2(x_1) - \min_{\mathbb{T}^N} \phi_1 \le \epsilon\phi_1(x_1) + H_1(x_1, 0) \le 2\mathcal{H}.$$

Taking into account (4.4), we get the desired inequality. □

3.2. **Strong maximum principle for systems.** The following extension of the strong maximum principle to parabolic systems is a crucial ingredient in the proof of the large time behavior.

Theorem 3.4. *Assume that, for $i = 1, 2$, $A_i(x) \ge \nu_i(x)I$ with $\nu_i \in C(\mathbb{T}^N)$, $\nu_i \ge 0$ and*

$$\text{for all } x \in \mathbb{T}^N, \ \sum_{i=1,2} \nu_i(x) > 0. \tag{3.9}$$

If u is a continuous viscosity subsolution of

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \mathrm{trace}(A_1(x)D^2 u_1) - C|Du_1| + u_1 - u_2 = 0, & x \in \mathbb{T}^N \times (0, +\infty), \\ \dfrac{\partial u_2}{\partial t} - \mathrm{trace}(A_2(x)D^2 u_2) - C|Du_2| + u_2 - u_1 = 0, & \\ u_i(x, 0) = u_{0i}(x), & x \in \mathbb{T}^N, \end{cases} \tag{3.10}$$

which attains a maximum at $(\overline{x}, \overline{t}) \in \mathbb{T}^N \times (0, +\infty)$, then u is constant.

Remark 3.5.
(i) A new feature of the above result is that one only need a partial nondegeneracy condition for the diffusion matrices A_i in the following sense. At each point of $\mathbb{T}^N$, there exists at least one equation such that $A_i(x)$ is nondegenerate. It can be interpreted using optimal control as follows. When considering the stochastic control problem associated with the equation in (3.10), it means that the controlled process visits any open set of $\mathbb{T}^N$ almost surely for any open time interval (see Section 4.4 for the control interpretation).
(ii) This result contains, as a particular case, stationary systems.

Proof of Theorem 3.4.
1. Suppose that

$$M := \sup_{(x,t) \in \mathbb{T}^N \times [0,+\infty),\, j=1,2} u_j(x, t) = u_i(\overline{x}, \overline{t}), \quad \overline{t} > 0. \tag{3.11}$$

We do a formal calculation. A rigorous one can be made using classical viscosity techniques. At the point $(\overline{x}, \overline{t})$, we have $\frac{\partial u_i}{\partial t} = 0$, $Du_i = 0$ and $-D^2 u_i \ge 0$. From the ith equation, we obtain

$$u_i(\overline{x}, \overline{t}) - u_{i+1}(\overline{x}, \overline{t}) \le 0, \quad \text{where we identify } u_3 = u_1.$$

Therefore $u_1(\overline{x}, \overline{t}) = u_2(\overline{x}, \overline{t})$. It follows that $(\overline{x}, \overline{t})$ is the common maximum point of u_1, u_2 in $\mathbb{T}^N \times [0, +\infty)$.
2. By (3.9), for all $x \in \mathbb{T}^N$, there exists $j \in \{1, 2\}$ such that $\nu_j(x) > 0$. By continuity, there exists $r_x > 0$ such that $\overline{B}(x, r_x) \subset \Omega_j := \{x \in \mathbb{T}^N : \nu_j(x) > 0\}$. It follows

$$\mathbb{T}^N = \bigcup_{x \in \mathbb{T}^N} B(x, r_x)$$

and, by compactness, there exists a finite covering

$$\mathbb{T}^N = \bigcup_{p=1}^{n} B_p \quad \text{with, for all } p,\ \overline{B}_p \subset \Omega_j \text{ for some } j \in \{1,2\}. \tag{3.12}$$

It follows that there exist p, j such that $\overline{x} \in \overline{B}_p \subset \Omega_j$. By continuity of ν_j and compactness, we have

$$\inf_{y\in \overline{B}_p, |\xi|=1} \langle A_j(y)\xi, \xi\rangle \geq \inf_{y\in\overline{B}_p} \nu_j(y) =: \nu > 0.$$

Hence the jth equation is uniformly parabolic in B_p. Moreover, the maximum in (3.11) is also a maximum on $B_p \times [0,+\infty)$. We set $u_k^M = u_k - M \leq 0$ for all k. So we have

$$\begin{aligned}
& \frac{\partial u_j^M}{\partial t} - \mathrm{trace}(A_j D^2 u_j^M) - C|Du_j^M| + u_j^M \\
\leq\ & \frac{\partial u_j^M}{\partial t} - \mathrm{trace}(A_j D^2 u_j^M) - C|Du_j^M| + u_j^M - u_{j+1}^M \quad \text{identifying } u_3 = u_1, \\
=\ & \frac{\partial u_j}{\partial t} - \mathrm{trace}(A_j D^2 u_j) - C|Du_j| + u_j - u_{j+1} \leq 0 \quad \text{in } B_p.
\end{aligned}$$

By the strong maximum principle for viscosity solutions of single parabolic equations (see, e.g., [14]), we obtain $u_j^M \equiv 0$ in $B_p \times [0,+\infty)$. Coming back to Step 1, we infer that the maximum in (3.11) is achieved for $i = 1, 2$ at every $(y,t) \in B_p \times [0,+\infty)$.
3. By (3.12), there exist p' and x' such that $x' \in B_p \cap B_{p'}$. It follows that the u_j's achieve their maximum over $B_{p'} \times [0,+\infty)$ at $(x', \overline{t})$. Repeating Step 2, we conclude that the u_j's are constant in $B_{p'} \times [0,+\infty)$. From (3.12), we conclude that $u_j \equiv M$ for all j and $(x,t) \in \mathbb{T}^N \times [0,+\infty)$. □

3.3. **Ergodic problem.** The uniform gradient bound established for (3.1) allows us to solve the ergodic problem.

The following assumption is used to "linearize" the system in order to apply the strong maximum principle-Theorem 3.4.

$$H_i \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{T}^N \times \mathbb{R}^N). \tag{3.13}$$

Theorem 3.6 (Ergodic problem). *Suppose that the assumptions of Theorem 3.1 hold for* (3.1)*. Then, there exists a solution* $(c, v) \in \mathbb{R}^2 \times W^{1,\infty}(\mathbb{T}^N)^2$ *of* (1.4)*. The ergodic constant* c *is unique and* $c = (c_1, c_1)$*. If, in addition, the* H_i*'s satisfy* (3.13)*, then* v *is unique up to an additive constant vector.*

For $m = 1$, we find the classical results for scalar equations (see [24, 7]).

Proof of Theorem 3.6. Let ϕ^ϵ be the continuous solution of (3.1) given by Theorem 3.1. We first claim that there exists a constant C' independent of ϵ such that

$$\left|\phi_i^\epsilon(x) - \phi_{i+1}^\epsilon(x)\right| \leq C', \quad x \in \mathbb{T}^N, i = 1,2 \text{ and we identify } \phi_3^\epsilon = \phi_1^\epsilon. \tag{3.14}$$

Indeed, choosing $\overline{x}_i$ as the maximum point of ϕ_i^ϵ, using the ith equation and (4.4), we have

$$|\max \phi_1^\epsilon - \max \phi_2^\epsilon| \leq C.$$

From Theorem 3.1, we obtain

$$\begin{aligned}\phi_i^\epsilon(x)-\phi_{i+1}^\epsilon(x) &= [\max\phi_i-\max\phi_{i+1}]+[\phi_i^\epsilon(x)-\phi_i^\epsilon(\overline{x}_i)]+[\phi_{i+1}^\epsilon(\overline{x}_{i+1})-\phi_{i+1}^\epsilon(x)]\\ &\leq C+\mathrm{osc}(\phi_1)+\mathrm{osc}(\phi_2).\end{aligned}$$

The lower bound is established in the same way by introducing $\underline{x}_i$ as the minimum point of ϕ_i.

Fix $x^*\in\mathbb{T}^N$ and set $\rho_i^\epsilon:=\phi_i^\epsilon(x^*)-\phi_{i+1}^\epsilon(x^*)$ and $v_i^\epsilon(x):=\phi_i^\epsilon(x)-\phi_i^\epsilon(x^*)$ for all $i=1,2$. From Theorem 3.1, (4.4) and (3.14), using Ascoli-Arzela's theorem, there exists a sequence $\epsilon_k\to 0$ so that

$$v_i^{\epsilon_k}\to v_i,\qquad \epsilon_k\phi_i^{\epsilon_k}\to -c_i,\qquad \rho_i^{\epsilon_k}\to\rho_i,$$

uniformly on $\mathbb{T}^N$ as $k\to\infty$ for some $v_i\in W^{1,\infty}(\mathbb{T}^N)$, $c_i,\rho_i\in\mathbb{R}$. Note that c_i is a constant independent of x^* while v_i,ρ_i depend on x^*.

Now, we rewrite (3.1) as

$$\left\{\begin{array}{l}\epsilon\phi_1^\epsilon-\mathrm{trace}(A_1(x)D^2v_1^\epsilon)+H_1(x,Dv_1^\epsilon)+v_1^\epsilon-v_2^\epsilon+\rho_1^\epsilon=0,\\ \epsilon\phi_2^\epsilon-\mathrm{trace}(A_2(x)D^2v_2^\epsilon)+H_2(x,Dv_2^\epsilon)+v_2^\epsilon-v_1^\epsilon+\rho_2^\epsilon=0.\end{array}\right.\tag{3.15}$$

Passing to the limit along the subsequence ϵ_k, by stability, we obtain that $v=(v_1,v_2)\in W^{1,\infty}(\mathbb{T}^N)^2$ is solution of

$$\left\{\begin{array}{l}-\mathrm{trace}(A_1(x)D^2v_1)+H_1(x,Dv_1)+v_1-v_2+\rho_1=c_1,\\ -\mathrm{trace}(A_2(x)D^2v_2)+H_2(x,Dv_2)+v_2-v_1+\rho_2=c_2.\end{array}\right.$$

Multiplying (3.15) by ϵ and passing to the limit along the subsequence ϵ_k, we get $c_1=c_2$. Note that $\rho_1^\epsilon+\rho_2^\epsilon=0$, so $\rho_1=-\rho_2$. We conclude by setting $\tilde{v}_1(x)=v_1(x)+\rho_1,\tilde{v}_2=v_2(x)$.

We finally mention that we can easily see the uniqueness of the ergodic constant by the comparison principle for (1.2) (Theorem 4.3). We claim that the uniqueness up to constant of solutions comes from the strong maximum principle (Theorem 3.4). Indeed, let $v,\tilde{v}$ be Lipschitz continuous solutions of (1.4) (the constant c is the same by the above). Since $|Dv|_\infty,|D\tilde{v}|_\infty\leq K$ for some K and $H_i\in W^{1,\infty}(\overline{B}(0,K))$, classical arguments in viscosity solutions imply that $v-\tilde{v}$ is a viscosity subsolution of the stationary version of (3.10) in $\mathbb{T}^N$. Therefore $v=\tilde{v}+C$ where $C\in\mathbb{R}^m$ is a constant. □

3.4. **Large time behavior result.** We first give a general result and then apply it to asymmetric systems.

Theorem 3.7. *(Large time behavior) Suppose that* (3.9), (3.13) *hold. Suppose there exists a viscosity solution* $u\in C(\mathbb{T}^N\times[0,+\infty))^2$ *of* (1.2)*, a solution* $((c_1,c_1),v)\in\mathbb{R}^2\times W^{1,\infty}(\mathbb{T}^N)^2$ *of the ergodic problem* (1.4) *and* $K>0$ *such that*

$$|Du_i(\cdot,t)|_\infty,\leq K,\quad t\geq 0, i=1,2.$$

Then, there exists a constant $\ell\in\mathbb{R}$ *such that*

$$u(x,t)+(c_1,c_1)t+(\ell,\ell)\to v(x)\quad uniformly\ as\ t\to+\infty.$$

Proof of Theorem 3.7. Several parts of this proof are inspired by [7].
1. Since $|Du_i(\cdot,t)|_\infty\leq K$, we deduce from Theorem 4.3 that u is the unique continuous viscosity solution of (1.2).

Defining $c = (c_1, c_1)$, we see that $v^{\pm}(x,t) = v(x) - ct \pm (C, C)$ is a super and subsolution, respectively, of (1.2) for C large enough. Thanks to the comparison principle, we have

$$v_i(x) - C \le u_i(x,t) + ct \le v_i(x) + C, \quad x \in \mathbb{T}^N, i = 1, 2.$$

It follows that $u_i(x,t) + ct$ is bounded; by the change of function $u \to u + ct$, we may assume without loss of generality that u is bounded and the ergodic constant c is 0.

Set $m(t) = \max_{x\in\mathbb{T}^N,\, i=1,2}\{u_i(x,t) - v_i(x)\}$. The comparison principle claims that m is nonincreasing and hence, $m(t) \to \ell$ as $t \to \infty$.

2. A by-product of Step 1 is that $\{u(\cdot, t), t > 0\}$ is relatively compact in $W^{1,\infty}(\mathbb{T}^N)^2$. So we can extract a sequence, $t_j \to +\infty$ such that $u(\cdot, t_j) \to \overline{u} \in W^{1,\infty}(\mathbb{T}^N)^2$. Applying Theorem 4.3 for (1.2), we obtain

$$|u_i(x, t+t_p) - u_i(x, t+t_q)| \le \max_{y\in\mathbb{T}^N,\, j=1,2} |u_j(y, t_p) - u_j(y, t_q)|, \quad x \in \mathbb{T}^N,\ t \ge 0,\ p, q \in \mathbb{N},$$

which proves that $(u(\cdot, \cdot + t_p))_p$ is a Cauchy sequence in $C(\mathbb{T}^N \times [0, +\infty))^2$. We call u_∞ its limit. Notice, on the one hand, that $|Du_{\infty i}(\cdot, t)|_\infty \le K$ for all i, t and, on the other hand, by stability, u_∞ is solution of (1.2) with initial data $\overline{u}$.

3. Using the uniform convergence of $u_i(., t + t_j)$, we pass to the limit with respect to j in

$$m(t + t_j) = \max_{i,x}(u_i(x, t+t_j) - v_i(x))$$

to obtain $\ell = \max_{i,x}(u_{\infty i}(x,t) - v_i(x))$ for any $t > 0$.

4. Since u_∞ is solution of (1.2) and v is solution of (1.4) and, up to increase K, both are K-Lipschitz continuous in x, thanks to the Lipschitz continuity of H_i with respect to p, see (3.13), we obtain that $u_\infty - v$ is subsolution of (3.10). The strong maximum principle (Theorem 3.4), then implies

$$\ell = u_{\infty i}(x,t) - v_i(x) \quad (x,t) \in \mathbb{T}^N \times [0, +\infty), i = 1, 2.$$

Noticing that $(\ell, \ell) + v(x)$ does not depend on the choice of subsequences, we obtain $u_i(x,t) - \ell - v_i(x) \to 0$ uniformly in x as $t \to \infty$, for $i = 1, 2$. □

We apply the previous result for the particular systems we studied in Section 3.

Corollary 3.8. *(Large time behavior) Assume* (3.9)*,* (3.13) *and suppose that the assumptions of Theorem 3.1 are in force. Then, for any initial condition* $u_0 \in W^{1,\infty}(\mathbb{T}^N)^2$*,*

- (i) *The system* (1.2) *has a unique viscosity solution* $u \in C(\mathbb{T}^N \times [0, +\infty))^2$ *and there exists* $K > 0$ *such that* $|Du_i(\cdot, t)|_\infty \le K$ *for all* $t \ge 0, i = 1, 2$.
- (ii) *There exist a unique ergodic constant* $c = (c_1, c_1) \in \mathbb{R}$ *and* $v \in W^{1,\infty}(\mathbb{T}^N)^2$ *solution of* (1.4) *such that* $u(x,t) + ct \to v(x)$ *uniformly as* $t \to +\infty$.

Remark 3.9. Remember that the classical assumptions on the Hamiltonians in (1.2) do not hold in order to have comparison and uniqueness of viscosity solutions. It is why, we prove first the existence of a (Lipschitz) continuous in space viscosity solution of (1.2). To this purpose, we need the initial condition to be Lipschitz continuous though this regularity assumption is not necessary to obtain the long time behavior.

Proof of Corollary 3.8.
1. We introduce the truncated evolutive system (1.2) in $\mathbb{T}^N \times [0, +\infty)$ with the H_{in} defined by (4.5). From Theorem 4.3 and Perron's method, there exists a unique continuous viscosity solution u_n in $\mathbb{T}^N \times [0, +\infty)$.

In the following, M is a constant which may vary line to line but is independent of n.
2. Applying Theorem 3.6 with the H_{in}'s, we obtain the existence of $(c_n, v_n) \in \mathbb{R}^2 \times W^{1,\infty}(\mathbb{T}^N)^2$. Since the H_{in}'s satisfy (3.3)-(3.4) with constants independent of n for large n, and $c_n \to c$ when $n \to +\infty$, we obtain from Theorem 3.1 that $|v_n|_\infty, |Dv_n|_\infty \leq M$.
3. Noticing that $v_n(x) - c_n t$ satisfies (1.2) with the H_{in}'s, by Theorem 4.3, we obtain

$$v_{ni}(x) - M \leq u_{ni}(x,t) + c_{ni}t \leq v_{ni}(x) + M, \quad (x,t) \in \mathbb{T}^N \times [0,+\infty), i = 1, 2.$$

Therefore $|u_n(\cdot,t) + c_n t|_\infty, \mathrm{osc}(u_n(\cdot,t)) \leq M$.
4. We claim that $|Du_n(\cdot,t)|_\infty \leq M$. To prove this fact, we repeat the proof of Lemma 3.3 for the evolutive system (1.2) with the H_{in}'s. Let $T > 0$ and consider

$$\max_{(x,y)\in(\mathbb{T}^N)^2, t\in[0,T], 1\leq i\leq m} \{w_{ni}(x,t) - w_{ni}(y,t) - \psi(|x-y|)\},$$

where $e^{w_{ni}(x,t)} = u_{ni}(x,t) - \min_{\mathbb{T}^N} u_{ni}(\cdot,t) + 1$, and ψ is given by (2.10). If the maximum is nonpositive we are done. Otherwise it is positive and achieved at some $(\overline{x}, \overline{y}, \overline{t}, i)$ with $\overline{x} \neq \overline{y}$ and we can write the viscosity inequalities for the ith equation, see [13, Theorem 8.3]: For every $\varrho > 0$, there exist $(a,p,X) \in \overline{P}^{2,+} w_{ni}(\overline{x},\overline{t})$ and $(b,p,Y) \in \overline{P}^{2,-} w_{ni}(\overline{y},\overline{t})$ such that (2.16) holds since $a - b = 0$. We achieve a contradiction as in the proof of Lemma 3.3.
5. From Ascoli-Arzela's theorem and the stability result for systems, by letting $n \to +\infty$, we obtain a continuous viscosity u solution of (1.2) in $\mathbb{T}^N \times [0,+\infty)$, with $|Du(\cdot,t)|_\infty \leq M$. This solution is unique thanks to Theorem 4.3. It ends the proof of (i). The proof of (ii) is an immediate consequence of Theorems 3.6 and 3.7. □

3.5. **Examples and extensions.** We give some examples such that our convergence result, Corollary 3.8, holds.

Example 3.10. (Systems with possibly degenerate equations and superlinear Hamiltonians) Consider

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - d_1(x)\Delta u_1 + a_1(x)|Du_1|^{1+\alpha_1} + u_1 - u_2 = f_1(x), & x \in \mathbb{T}^N \times (0,+\infty), \\ \dfrac{\partial u_2}{\partial t} - d_2(x)\Delta u_2 + a_2(x)|Du_2|^{1+\alpha_2} + u_2 - u_1 = f_2(x), & x \in \mathbb{T}^N \times (0,+\infty), \\ u_1(x,0) = u_{01}(x),\ u_2(x,0) = u_{02}(x), & x \in \mathbb{T}^N, \end{cases}$$

where $d_i \geq 0, a_i, f_i, u_{0i}$ are Lipschitz continuous, $\alpha_i > 0$ and $a_i > 0$. Suppose moreover that for any $x \in \mathbb{T}^N$, we have either $d_1(x) > 0$ or $d_2(x) > 0$ (this implies (3.9)). So, the hypotheses of Corollary 3.8 hold.

When there is at least one sublinear Hamiltonian, we need that the diffusion matrix of the corresponding equation is uniformly elliptic. But we permit other diffusion matrices to be degenerate everywhere.

Example 3.11. (Asymmetric systems with degenerate equations) In the following system, the first equation is of sublinear type while the second is of superlinear type,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \mathrm{trace}(A_1 D^2 u_1) + \sup_{\theta\in\Theta}\{-\langle b_{\theta 1}(x), Du_1\rangle - f_{\theta 1}(x)\} + u_1 - u_2 = f_1(x), \\ \dfrac{\partial u_2}{\partial t} - \mathrm{trace}(A_2 D^2 u_2) + a_2(x)|Du_2|^{1+\alpha} + u_2 - u_1 = f_2(x), \quad x \in \mathbb{T}^N \times (0,+\infty), \\ u_1(x,0) = u_{01}(x),\ u_2(x,0) = u_{02}(x), \quad x \in \mathbb{T}^N, \end{cases} \tag{3.16}$$

where the functions $b_{\theta 1}, f_{\theta 1}, a_2, f_2, u_{01}, u_{02}$ satisfy

$$|f(x)| \leq C, \quad |f(x) - f(y)| \leq C|x-y|, \qquad x, y \in \mathbb{T}^N, \theta \in \Theta \tag{3.17}$$

and $\alpha > 0$ and $a_2(x) > 0$. If we suppose moreover that $A_1 > 0$ in $\mathbb{T}^N$ (notice that we only assume $A_2 \geq 0$), then the hypotheses of Corollary 3.8 hold.

We end with some possible extensions to fully nonlinear systems with $m \geq 2$ equations of the form (1.1) with $H_{\theta i} = \underline{H}_{\theta i} + \overline{H}_{\theta i}$. We assume that the system is monotone and irreducible, i.e., the coupling matrix $D = (d_{ij})_{1 \leq i,j \leq m}$ satisfies

$$d_{ii} \geq 0, \quad d_{ij} \leq 0 \text{ for } i \neq j \quad \text{and} \quad \sum_{j=1}^{m} d_{ij} = 0 \text{ for all } i,$$

for all subset $\mathcal{I} \subsetneq \{1, \cdots, m\}$, there exists $i \in \mathcal{I}$ and $j \notin \mathcal{I}$ such that $d_{ij} \neq 0$.

In this case, it is possible to find $\Lambda \in \mathbb{R}^m$ with positive components $\Lambda_i > 0$ such that $D^T \Lambda = 0$ ([9]) and to define $\lambda_D := \max_{1 \leq i \leq m} \frac{1}{\Lambda_i} \sum_{j \neq i} \Lambda_j$. Then it is possible to generalize the previous results: We assume that $A_{\theta i}$ satisfies (2.43) and $\underline{H}_{\theta i}, \overline{H}_{\theta i}$ satisfy (2.44).

- Theorem 3.1 holds if, for each $1 \leq i \leq m$, the ith-equation satisfies, uniformly with respect to θ, either (3.3) or (3.4) with $|\overline{H}(\cdot, 0)|$ replaced by $(2\lambda_D + 1)\mathcal{H}$ where $\mathcal{H}$ is defined in (3.2) with a supremum over all $1 \leq j \leq m$, $\theta \in \Theta$. Notice that, when $m = 1$ (resp. $m = 2$), $\lambda_D = 0$ (resp. $\lambda_D = 1$) and we recover exactly Theorem 2.6 (resp. Theorem 3.1).
- Theorem 3.6 holds under the assumptions above and (3.13) uniformly with respect to θ.
- Corollary 3.8 holds under the assumptions above and (3.9) with $A_{\theta i} \geq \nu_i I$.

4. Appendix

4.1. **Proof of Lemma 2.4.** From (2.13), for every $\zeta, \xi \in \mathbb{R}^N$, we have

$$\langle X\zeta, \zeta \rangle - \langle Y\xi, \xi \rangle \leq \Psi' \langle \zeta - \xi, B(\zeta - \xi) \rangle + \Psi'' \langle \zeta - \xi, (q \otimes q)(\zeta - \xi) \rangle + O(\varrho).$$

We estimate $\mathrm{trace}(A(\overline{x})X)$ and $\mathrm{trace}(A(\overline{y})Y)$ using two orthonormal bases $(e_1, \cdots, e_N)$ and $(\tilde{e}_1, \cdots, \tilde{e}_N)$ in the following way:

$$\begin{aligned} T := \mathrm{trace}(A(\overline{x})X - A(\overline{y})Y) &= \sum_{i=1}^{N} \langle X\sigma(\overline{x})e_i, \sigma(\overline{x})e_i \rangle - \langle Y\sigma(\overline{y})\tilde{e}_i, \sigma(\overline{y})\tilde{e}_i \rangle \\ &\leq \sum_{i=1}^{N} \Psi' \langle \zeta_i, B\zeta_i \rangle + \Psi'' \langle \zeta_i, (q \otimes q)\zeta_i \rangle + O(\varrho) \\ &\leq \Psi'' \langle \zeta_1, (q \otimes q)\zeta_1 \rangle + \sum_{i=1}^{N} \Psi' \langle \zeta_i, B\zeta_i \rangle + O(\varrho), \end{aligned} \tag{4.1}$$

where we set $\zeta_i = \sigma(\overline{x})e_i - \sigma(\overline{y})\tilde{e}_i$ and noticing that $\Psi'' \langle \zeta_i, (q \otimes q)\zeta_i \rangle = \Psi'' \langle \zeta_i, q \rangle^2 \leq 0$ since Ψ is concave.

We now build suitable bases in two following cases. In the case where σ is degenerate, we choose any orthonormal basis such that $e_i = \tilde{e}_i$. It follows

$$\begin{aligned} T &\leq \sum_{i=1}^{N} \Psi' \langle (\sigma(\overline{x}) - \sigma(\overline{y}))e_i, B(\sigma(\overline{x}) - \sigma(\overline{y}))e_i \rangle + O(\varrho) \\ &\leq \Psi' N |\sigma(\overline{x}) - \sigma(\overline{y})|^2 |B| + O(\varrho) \\ &\leq \Psi' N |\sigma_x|_\infty^2 |\overline{x} - \overline{y}| + O(\varrho) \end{aligned}$$

from (2.2) and since $|B| \leq 1/|\overline{x} - \overline{y}|$.

When (2.4) holds, i.e., $A(x) \geq \nu I$ for every x, we can set

$$e_1 = \frac{\sigma(\overline{x})^{-1} q}{|\sigma(\overline{x})^{-1} q|}, \quad \tilde{e}_1 = -\frac{\sigma(\overline{y})^{-1} q}{|\sigma(\overline{y})^{-1} q|}, \quad \text{where } q \text{ is given by (2.14).}$$

If e_1 and $\tilde{e}_1$ are collinear, then we complete the basis with orthogonal unit vectors $e_i = \tilde{e}_i \in e_1^\perp$, $2 \leq i \leq N$. Otherwise, in the plane $\mathrm{span}\{e_1, \tilde{e}_1\}$, we consider a rotation $\mathcal{R}$ of angle $\frac{\pi}{2}$ and define

$$e_2 = \mathcal{R} e_1, \quad \tilde{e}_2 = -\mathcal{R} \tilde{e}_1.$$

Finally, noticing that $\mathrm{span}\{e_1, e_2\}^\perp = \mathrm{span}\{\tilde{e}_1, \tilde{e}_2\}^\perp$, we can complete the orthonormal basis with unit vectors $e_i = \tilde{e}_i \in \mathrm{span}\{e_1, e_2\}^\perp$, $3 \leq i \leq N$.

From (2.4), we have

$$\nu \leq \frac{1}{|\sigma(x)^{-1} q|^2} \leq |\sigma|_\infty^2 \leq C^2. \tag{4.2}$$

It follows

$$\langle \zeta_1, (q \otimes q) \zeta_1 \rangle = \left(\frac{1}{|\sigma(\overline{x})^{-1} q|} + \frac{1}{|\sigma(\overline{y})^{-1} q|} \right)^2 \geq 4\nu.$$

From (2.14), we deduce $Bq = 0$. Therefore

$$\langle \zeta_1, B\zeta_1 \rangle = 0.$$

For $3 \leq i \leq N$, using (2.2),

$$\langle \zeta_i, B\zeta_i \rangle = \langle (\sigma(\overline{x}) - \sigma(\overline{y}))e_i, B(\sigma(\overline{x}) - \sigma(\overline{y}))e_i \rangle \leq |\sigma_x|_\infty^2 |\overline{x} - \overline{y}| \leq \sqrt{N} C^2.$$

since $|B| \leq 1/|\overline{x} - \overline{y}|$ and $|\overline{x} - \overline{y}| \leq \sqrt{N}$. We have

$$|\zeta_2| = |(\sigma(\overline{x}) - \sigma(\overline{y}))\mathcal{R} e_1 + \sigma(\overline{y}) \mathcal{R}(e_1 + \tilde{e}_1)| \leq C|\overline{x} - \overline{y}| + C|e_1 + \tilde{e}_1|.$$

It remains to estimate

$$\begin{aligned} |e_1 + \tilde{e}_1| &\leq \frac{1}{|\sigma_\theta(\overline{x})^{-1} q|} |\sigma(\overline{x})^{-1} q - \sigma(\overline{y})^{-1} q| + |\sigma \overline{y})^{-1} q| \left| \frac{1}{|\sigma(\overline{x})^{-1} q|} - \frac{1}{|\sigma(\overline{y})^{-1} q|} \right| \\ &\leq \frac{2|\sigma|_\infty |\sigma_x|_\infty^2}{\nu} |\overline{x} - \overline{y}| = \frac{2C^3}{\nu} |\overline{x} - \overline{y}|, \end{aligned}$$

from (4.2) and $|(\sigma^{-1})_x|_\infty \leq |\sigma_x|_\infty^2 / \nu$.

From (4.1), we finally obtain the conclusion $T \leq 4\nu \Psi'' + \tilde{C} \Psi' + O(\varrho)$ where

$$\tilde{C} = \tilde{C}(N, \nu, |\sigma|_\infty, |\sigma_x|_\infty) := C^2 \sqrt{N} (N - 2 + (1 + \frac{2C^3}{\nu})^2) \text{ with } C := \max\{|\sigma|_\infty, |\sigma_x|_\infty\}. \tag{4.3}$$

4.2. **Comparison principles for stationary systems.** The following results are stated in [7] in the case of a scalar equation without control. We state them for systems.

Theorem 4.1. *Let $\psi^\epsilon \in USC(\mathbb{T}^N)^2$ and $\phi^\epsilon \in LSC(\mathbb{T}^N)^2$ be respectively a viscosity subsolution and supersolution of* (1.12)*. Assume that either ψ^ϵ or $\phi^\epsilon \in W^{1,\infty}(\mathbb{T}^N)^2$. Then $\psi_i \leq \phi_i$ in $\mathbb{T}^N$ for $i = 1, 2$.*

A useful consequence of Theorem 4.1 is that, for any viscosity solution ϕ^ϵ of (1.12),

$$|\epsilon \phi_i^\epsilon|_\infty \leq \sup_{j=1,2} |H_j(\cdot, 0)|_\infty =: \mathcal{H}, \quad i = 1, 2. \tag{4.4}$$

Proof of Theorem 4.1. We only give a sketch of proof since it is classical. We suppose without loss of generality that $\sup_i |D\psi_i|_\infty =: L < \infty$. Set

$$M_0 = \max_{x \in \mathbb{T}^N,\, j=1,2} \{\psi_j(x) - \phi_j(x)\},$$

$$M_\alpha = \max_{x,y \in \mathbb{T}^N,\, j=1,2} \{\psi_j(x) - \phi_j(y) - \alpha^2 |x - y|^2\} = \psi_i(\overline{x}) - \phi_j(\overline{y}) - \alpha^2 |\overline{x} - \overline{y}|^2.$$

We have some classical estimates

$$\alpha^2 |\overline{x} - \overline{y}|^2 \underset{\alpha \to \infty}{\to} 0, \quad |p| \leq L \text{ where } p = 2\alpha^2 (\overline{x} - \overline{y}), \quad \limsup_{\alpha \to \infty} M_\alpha = M_0.$$

After subtracting write the viscosity inequalities at $(\overline{x}, \overline{y})$ of (1.12), we get

$$\epsilon(\psi_i(\overline{x}) - \phi_i(\overline{y})) - \mathrm{trace}(A_i(\overline{x})X - A_i(\overline{y})Y) + H_i(\overline{x}, p) - H_i(\overline{y}, p)\}$$
$$+ [\psi_i(\overline{x}) - \phi_i(\overline{y})] - [\psi_{i+1}(\overline{x}) - \phi_{i+1}(\overline{y})] \leq 0.$$

Since $\psi_j(\overline{x}) - \phi_j(\overline{y}) \leq \psi_i(\overline{x}) - \phi_i(\overline{y})$, we get $[\psi_i(\overline{x}) - \phi_i(\overline{y})] - [\psi_{i+1}(\overline{x}) - \phi_{i+1}(\overline{y})] \geq 0$. Moreover $-\mathrm{trace}(A_i(\overline{x})X - A_i(\overline{y})Y) \geq o_\alpha(1)$. Therefore $\epsilon(\psi_i(\overline{x}) - \phi_i(\overline{y})) \leq o_\alpha(1)$, this yields $M_0 \leq 0$ as desired. □

In the proofs of Theorems 2.2 and 3.1, we need a discontinuous comparison result for truncated system when H_i in (1.12) is replaced by

$$H_{in} := \begin{cases} H_i(x, p) & \text{if } |p| \leq n, \\ H_i(x, n\frac{p}{|p|}) & \text{if } |p| \geq n. \end{cases} \tag{4.5}$$

Due to the truncation in the gradient variable, we do not need to assume that the sub- or supersolution is Lipschitz continuous. The proof follows the same lines as above. We obtain

Theorem 4.2. *Let $\psi \in USC(\mathbb{T}^N)^2$ and $\phi \in LSC(\mathbb{T}^N)^2$ be respectively a viscosity subsolution and supersolution of* (1.12) *with H_{in} defined by* (4.5)*. Then $\psi_i \leq \phi_i$ in $\mathbb{T}^N$ for all i.*

4.3. **Comparison for the evolutive problem.**

Theorem 4.3. *Let $u \in USC(\mathbb{T}^N \times [0, +\infty))^2$ and $v \in LSC(\mathbb{T}^N \times [0, +\infty))^2$ be respectively a viscosity subsolution and supersolution of* (1.2)*. Assume that either $u(\cdot, t)$ or $v(\cdot, t) \in W^{1,\infty}(\mathbb{T}^N)^2$ uniformly in $t \in [0, +\infty)$. Then*

$$u_i(x, t) - v_i(x, t) \leq \sup_{y \in \mathbb{T}^N, j=1,2} (u_j(y, 0) - v_j(y, 0))^+, \ (x, t) \in \mathbb{T}^N \times [0, +\infty), \quad i = 1, 2. \tag{4.6}$$

Theorem 4.4. *Let $u \in USC(\mathbb{T}^N \times [0, +\infty))^2$ and $v \in LSC(\mathbb{T}^N \times [0, +\infty))^2$ be respectively a viscosity subsolution and supersolution of* (1.2) *with H_{in} defined by* (4.5)*. Then* (4.6) *holds.*

The proofs are easy adaptations of the proofs of Theorems 4.1 and 4.2 in the case of degenerate parabolic systems.

4.4. **A Control-theoretic interpretation for** (3.16)**.** We can give an interpretation of weakly coupled systems as dynamic programming equations of hybrid systems with pathwise stochastic trajectories with random switching, see [18].

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, W_t be a $\mathcal{F}_t$-adapted standard N-Brownian motion such that $W_0 = 0$ a.s. Consider the controlled random evolution process (X_t, ν_t) with dynamics

$$\begin{cases} dX_t = b_{\theta_t \nu_t}(X_t)dt + \sqrt{2}\,\sigma_{\nu_t}(X_t)dW_t, \quad t > 0, \\ (X_0, \nu_0) = (x, i) \in \mathbb{T}^N \times \{1, 2\}, \end{cases}$$

where the control law $\theta_t : [0, \infty) \to \Theta$ is a measurable function.

For every θ_t there exists a $\mathcal{F}_t$-adapted solution (X_t, ν_t), where $X_t : [0, \infty) \to \mathbb{T}^N$ is piecewise C^1 and ν_t is a continuous-time Markov chain with state space $\{1, 2\}$ and probability transitions given by

$$\begin{aligned} \mathbb{P}\{\nu_{t+\Delta t} = 2 \,|\, \nu_t = 1\} = \Delta t + o(\Delta t), \\ \mathbb{P}\{\nu_{t+\Delta t} = 1 \,|\, \nu_t = 2\} = \Delta t + o(\Delta t). \end{aligned} \tag{4.7}$$

We introduce the value functions of the optimal control problems

$$u_i(x, t) = \inf_{\theta_t \in L^\infty([0,t],\Theta)} \mathbb{E}_{x,i}\{\int_0^t f_{\theta_s \nu_s}(X_s)ds + u_{0\nu_t}(X_t)\}, \quad i = 1, 2,$$

where $\mathbb{E}_{x,i}$ denote the expectation of a trajectory starting at x in the mode i.

The function $u = (u_1, u_2)$ satisfies the system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \mathrm{trace}(\sigma_1(x)\sigma_1(x)^T D^2 u_1) + \sup\limits_{\theta\in\Theta}\{-\langle b_{\theta 1}(x), Du_1\rangle - f_{\theta 1}(x)\} + u_1 - u_2 = 0 \\ \dfrac{\partial u_2}{\partial t} - \mathrm{trace}(\sigma_2(x)\sigma_2(x)^T D^2 u_2) + \sup\limits_{\theta\in\Theta}\{-\langle b_{\theta 2}(x), Du_2\rangle - f_{\theta 2}(x)\} + u_2 - u_1 = 0 \\ u_1(x, 0) = u_{01}(x), \;\; u_2(x, 0) = u_{02}(x). \end{cases}$$

By choosing $\Theta = \mathbb{R}^N$, setting $A_i = \sigma_i \sigma_i^T$, $i = 1, 2$ and assuming that $b_{\theta 1}, f_{\theta 1}, u_{01}, u_{02}$ satisfy (3.17), $b_{\theta 2} = 2\theta$ and $f_{\theta 2} = |\theta|^2 - f_2$ with f_2 continuous, we obtain (3.16) with $a_2 = 1$ and $\alpha = 1$. If moreover $A_1 > 0$, then the first equation is uniformly parabolic of sublinear type and the second one is possibly degenerate superlinear with a quadratic Hamiltonian. The assumptions of Theorems 3.1, 3.6 and 3.8 hold.

Roughly speaking, the Lipschitz regularization of u_1 is provided by the nondegenerate diffusion in mode 1 whatever the bounded drift does. While the Lipschitz regularity of u_2 comes from the controlable unbounded drift in mode 2 even if the diffusion degenerates. Assumption (3.9) is obviously satisfied since $\nu_1(x) := \min_{|\xi|=1}\langle A_1(x)\xi, \xi\rangle > 0$ for all $x \in \mathbb{T}^N$. So the strong maximum principle holds and we have the convergence when $t \to +\infty$. The point is that the nondegeneracy of A_1 implies also the convergence for u_2. This seems to be due to the combined effects of the nondegenerate Brownian motion in mode 1 together with the number of switchings which tends to $+\infty$ as $t \to +\infty$ (since the transition probabilities (4.7) are positive and independent of t). It follows that the process visits any open subset of $\mathbb{T}^N$ in each mode as $t \to +\infty$ yielding the convergence to an equilibrium state for (u_1, u_2).

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